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نزد گورنمنٹ کالج اصغر مال راولپنڈی
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CHAPTER: 2

Composition of Forces

Mechanics

Mechanics is the study of action of forces on bodies. It is divided into two branches.

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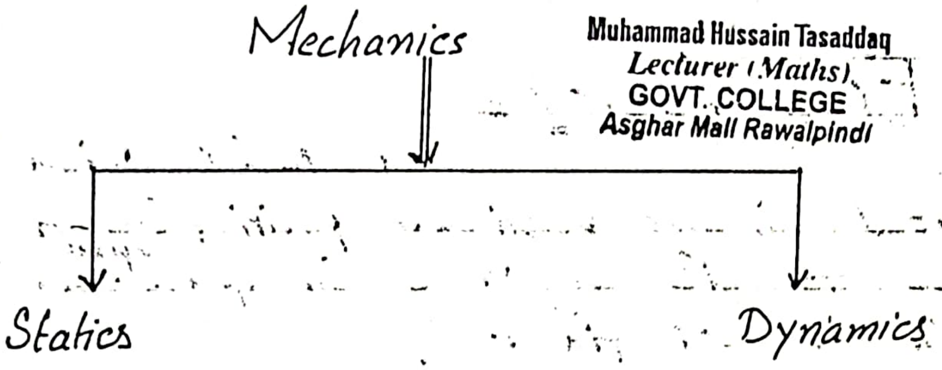
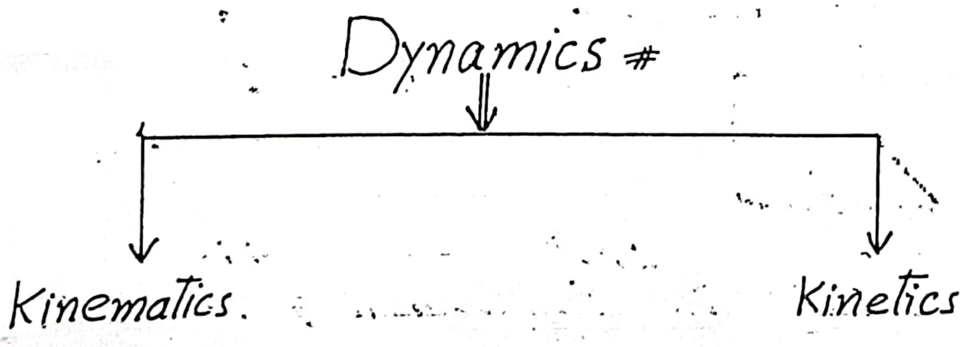


Statics # It deals with the state of rest of bodies or particle under the action of forces.

Note # The state of rest is a relative term and will be relative to some frame of reference or surroundings. The state of rest of a body relative to some frame of reference is called equilibrium.

Dynamics # It deals with bodies in motion under the action of forces.

It is further divided into two parts.



(i) Kinematics# It deals with the geometry of motion of bodies without considering the mass of body or force acting on the body i.e in kinematics we study the position of body, velocity at any instant and acceleration of body.

(ii) Kinetics# kinetics deals with the (forces) relations bet forces acting on the bodies and the resulting motion.

Free Vector#

A vector which is not located in any particular position i.e A vector free to lie anywhere in space or plane and its starting point is not fixed.

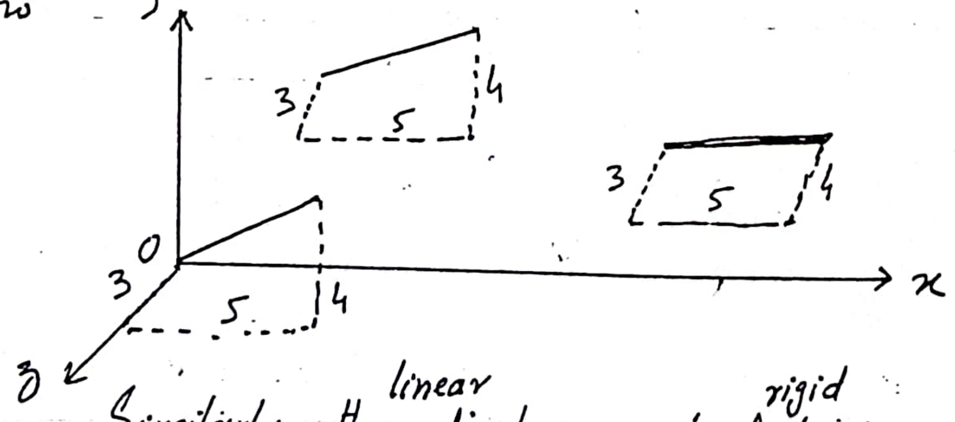
OR

A vector which can be shifted parallel to itself anywhere in the plane or space and its direction, magnitude remain same.

OR

A free vector is any one of a system of vectors which have common direction and magnitude but different origins (i.e position.)

e.g velocity vector $\underline{v} = 5\hat{i} + 4\hat{j} + 3\hat{k}$ can be represented by one of three vectors given below



Similarly the linear displacement of rigid body without rotation of rigid body can be represented by any one of the directed segments giving the displacements of its various points.

Thus a free vector can be represented by any one of a set of equal like parallel directed line Segments.

A physical quantity which is represented by a free vector is also called a free vector.

Bound OR Tied OR Localized Vector

A vector which can be represented by a unique directed segment i.e a bound vector has a fixed position. e.g position vector of point is a tied vector.

A physical quantity represented by a bound vector is also called bound vector.

A force acting on an elastic body is a bound vector because we can not in general change this force, by any displacement of directed segment representing it, without changing its effect.

Sliding Vector

A vector whose position can be changed along its line but its direction remain same.

OR

A vector which can be represented by any one of a system of directed segments obtained by sliding a directed segment along its line.

A physical quantity represented by a sliding vector is also called a sliding vector. e.g a force acting on a rigid body is a sliding vector because it may be applied at any point on its line of action without changing its effect.

Note In general by a vector we mean a free vector.

Characteristics of a Vector

A vector $\overrightarrow{AB}$ has the following characteristics.

(i) an origin or point of application (A)

(ii) a direction (defined analytically by three direction cosines of $\overrightarrow{AB}$ with respect to rectangular axes)

(iii) a magnitude (the length AB)

Particle #. A particle or a Mass Point is a body of negligible dimensions (i.e which can be ignored.

OR

A negligible mass concentrated at a point is called a particle.

OR

The smallest mass which can not be subdivided is called a particle.

The concept of particle in mechanics is similar to the concept of point in Geometry. Thus, a particle or a point mass must be located at some position.

Rigid Body #

A continuous collection of an infinite number of particles occupying some part of the space is called a rigid body or

Simply a body, if the distance between any two of its particles always remains constant. A rigid body always retains its shape and size i.e it can not be deformed.

Remarks# The above concepts of a particle and a rigid body are ideal concepts as they do not occur physically in actual practice.

Force#

Anything (any cause) that changes or tends to change the state of rest or motion of a body is called a force.

Generally, the forces are exerted when bodies come into contact with one another. But sometimes forces may act at a distance e.g the gravitational force, electric force and magnetic force.

Characteristics of a Force#

A force is a vector. Its characteristics are

(i) magnitude of force

(ii) direction of the force

(iii) The point of application of the force.

Line of Action of Force#

The line which passes through the point of application of force and is in the direction of force is called the line of action of force.

Note# The line of action of a force provides the direction of force but it must not be taken as the actual force vector (although sometimes it may be)

Force System # (A System of Forces)

A finite set of forces acting on a particle or a rigid body is called a force system or system of forces.

If a force system acts on a particle, then it is called a Concurrent force system.

A force system acting on a body or a particle is called Coplanar if the lines of action of all forces lie in a plane.

System

The system under consideration is something we pick out from the given arrangement in an arbitray menner.

Internal & External Forces

A force acting on a particle of a given system is called internal if it is exerted by another particle of the system; otherwise it is an external force.

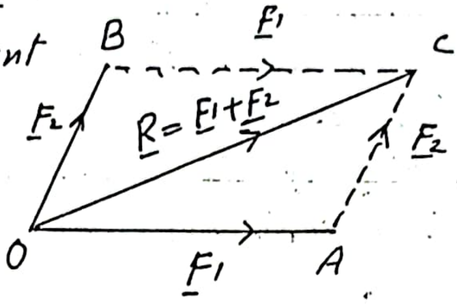
According to the law of action and reaction, internal foces occur in equal and opposite pairs and cancel each other's effect.

Fundmental Plane

The plane in which the system lies or to which the motion is parallel is called the fundamental plane.

Parallelogram Law of Forces

If two forces acting at a point are represented in magnitude and direction along the adjacent sides of the parallelogram and these sides pass through the point of application, then their resultant is represented by the diagonal of the parallelogram through the point of application.



In fig. resultant $\underline{R}$ is represented by diagonal OC of parallelogram OACB.

Behaviour of Force as Free, Bound and Sliding Vector

The force vector plays the roll of a free vector in the parallelogram law of forces.

Again as a force may depend on the (sys) position of the point of its application. This means that a force acting at particle behaves as a Bound Vector.

Finally, a force Vector acting on a rigid body acts like a sliding (force) because its effect will be observed everywhere on the line of action of force intersecting the body

Equilibrium of a Particle

A particle is in equilibrium if the resultant of all the forces acting on it is zero.

According to second law $\underline{F} = m\underline{a} = \frac{d}{dt}(m\underline{v})$

If the resultant force F acting on a particle is zero, then acceleration of particle is zero i.e either it is at rest or moves with uniform velocity. Also its linear momentum remains constant.

Thus the acceleration of particle in equilibrium is zero.

Note If the resultant of all forces acting on a particle is zero, then the forces are said to be in equilibrium.

Equilibrium of a Body

A body is in equilibrium when it has no acceleration, either of translation or rotation.

A rigid body is in equilibrium when its centre of mass has no acceleration and the body has no angular acceleration.

The conditions for a body to be in equilibrium are

(i) that the resultant of forces acting on it is zero

(ii) the the sum of the moments of these forces about every axis be equal to zero (about each of three mutually perpendicular axes sufficies)

Equilibrium of Forces (Balanced Force System)

Forces which has zero linear resultant and zero turning effect are said to be in equilibrium.

When such a force system is applied to a particle or body, it does not cause any change in the physical state (rest or uniform motion) of the particle or the body.

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Equilibrium Under the Action of Two Forces

If a System (body or particle) is in equilibrium under the action of two forces, then the forces have common line of action, equal magnitudes and opposite directions.

Equilibrium Under The action of Three Forces

If a System is in equilibrium under the action of three forces, then these forces are Coplanar and their Lines of action are either Concurrent or parallel.

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Law of Superposition # The effect of a given force system remains unaltered if any balanced force system is added to or deleted from it.

The Law of Transmissibility # The point of application of a force acting on a rigid body can be shifted to anyother point on the line of action of the force without changing the effect of force on the body. Thus a force acting on a rigid body behave as a sliding vector.

Note # A force acting on an elastic is a bound vector and its point of application can not in general be shifted along the line of action of force without changing the effect of force on the elastic body. Hence the Law of transmissibility applies only to a force acting on a rigid body.

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Equivalent Force Systems#

(of forces are equival)

Two systems of forces are equivalent if their effect on a body is the same in all respects. i.e the effect on body is unaltered if one system is replaced by another system.

The Resultant of a System of Forces#

of a set of

The resultant of a set of forces is the simplest possible equivalent system and the original set of forces is said to reduce to its resultant system.

Composition of Forces#

The reduction of a system of forces to the simplest possible equivalent system, is called the Composition of forces.

Concurrent Forces#

Forces whose lines of action intersect at one point are called Concurrent forces. OR Forces acting at a particle of body are called concurrent forces.

Note# The concurrent forces always have a resultant which also passes through the point of concurrency. This resultant can be found by two methods

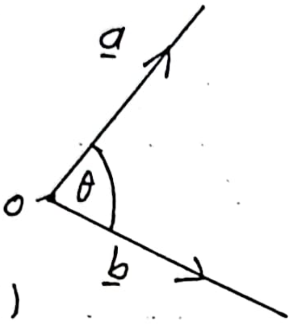
(a) Geometric Method

(b) Analytical Method

The Concept of Angle Between Vectors

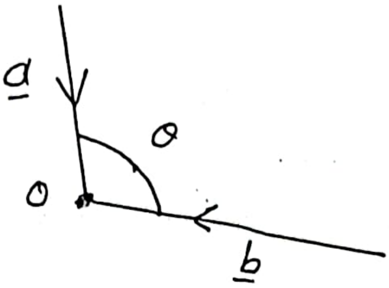
Let us understand. what is meant by the angle between two vectors.

Suppose that, from a point O, two line segments are drawn to represent vectors $\underline{a}$ and $\underline{b}$.



$\underline{a}$
O
θ
$\underline{b}$

(i)



$\underline{a}$
θ
O
$\underline{b}$

(ii)

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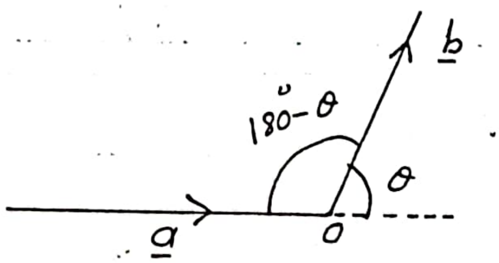
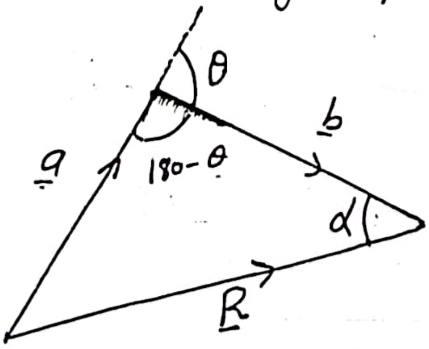


$\underline{b}$
$180^\circ-\theta$
θ
$\underline{a}$
O

(iii)

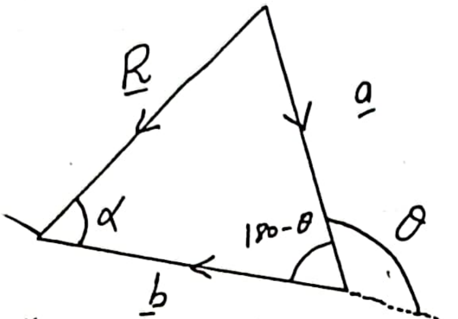
Then if the both vectors point away from O or both vectors point towards O as in figs (i) & (ii), the angle θ between the lines at O is the angle between the vectors. But if one vectors points toward O and one vector points away from O as in fig (iii) above, then the angles between lines at O is $(180^\circ - \theta)$, where θ is the angle between the vectors

Now vector triangle for fig (i) is.



θ
$\underline{b}$
$\underline{a}$
$180-\theta$
α
$\underline{R}$

Vector triangle for fig (ii) is



The magnitude R of the resultant can be found by using cosine formula in vector triangles.

$$R^2 = a^2 + b^2 - 2ab\cos(180-\theta)$$
$$= a^2+b^2+2ab\cos\theta$$
$$R = \sqrt{a^2+b^2+2ab\cos\theta}$$

The direction of the resultant can be determined by using the sine rule.

$$\frac{\sin\alpha}{a} = \frac{\sin(180^\circ-\theta)}{R}$$

$$\frac{\sin\alpha}{a} = \frac{\sin\theta}{R}$$

$$\sin\alpha = \frac{a\sin\theta}{R}$$

These formulae for calculating the values of R and α are valid whether θ is acute or obtuse.

We note that if $\underline{a}$ & $\underline{b}$ have equal magnitudes, then resultant is

$$R = \sqrt{a^2+a^2+2a^2\cos\theta} = \sqrt{2a^2(1+\cos\theta)}$$

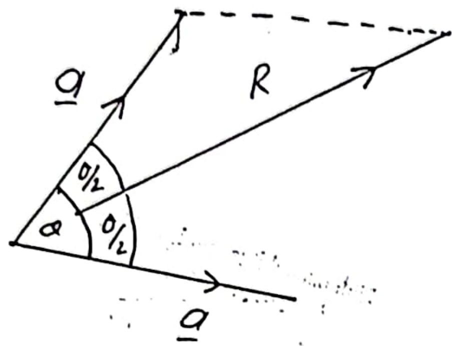
$$R = \sqrt{2a^2\cdot 2\cos^2(\theta/2)} = 2a\cos(\theta/2)$$

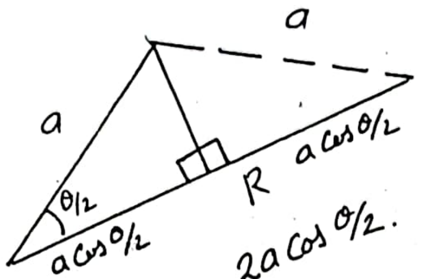
and

$$\sin\alpha = \frac{a\sin\theta}{2a\cos(\theta/2)} = \frac{2\sin\theta/2\cdot\cos\theta/2}{2\cos\theta/2}$$

$$\sin\alpha = \sin(\theta/2) \Rightarrow \alpha = \theta/2$$

i.e the resultant bisects the angle between the vectors



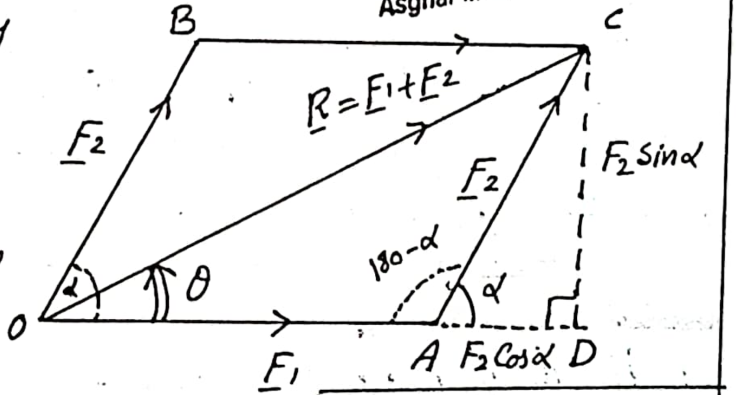

a
a
θ/2
R
a cos θ/2
a cos θ/2

$R = 2a\cos\theta/2.$

Resultant of Two Concurrent Forces#

Suppose two ~~forces~~ $\underline{F_1}$ & $\underline{F_2}$ act at point O and are represented by $\overrightarrow{OA}$, $\overrightarrow{OB}$. Let α be the angle between forces.

By parallelogram law of forces, their resultant $\underline{R}$ is given by


B
C
F₂
R = F₁+F₂
F₂
F₂ Sinα
α
θ
180-α
α
O
A
F₂Cosα
D
F₁

∵ OB is parallel to AC & OD cuts these

∴ $m\angle AOB = m\angle DAC$

$$\underline{R} = \underline{F_1} + \underline{F_2} \longrightarrow ①$$

i.e the resultant of two concurrent forces is given by their vector sum.

Magnitude of Resultant#

$$\underline{R}\cdot\underline{R} = (\underline{F_1}+\underline{F_2})\cdot(\underline{F_1}+\underline{F_2})$$

$$R^2 = \underline{F_1}\cdot\underline{F_2} + \underline{F_1}\cdot\underline{F_2} + \underline{F_2}\cdot\underline{F_1} + \underline{F_2}\cdot\underline{F_2}$$

$$= F_1^2 + 2\,\underline{F_1}\cdot\underline{F_2} + F_2^2$$

$$= F_1^2 + F_2^2 + 2F_1F_2\cos\alpha$$

$$\boxed{R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}} \longrightarrow ②$$

② gives the magnitude of the resultant.

OR

We can also find the magnitude of the resultant by applying Cosine rule to triangle OAC

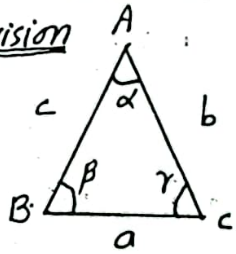
$$R^2 = F_1^2 + F_2^2 - 2F_1F_2\cos(180^\circ - \alpha)$$

$$= F_1^2 + F_2^2 + 2F_1F_2\cos\alpha$$

$$(\because \cos(180^\circ - \alpha) = -\cos\alpha)$$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}.$$

Revision



Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

Direction of The Resultant

Let the resultant $\underline{R}$ makes angle θ with force $\underline{F_1}$

Now

$$|\underline{F_1} \times \underline{R}| = |\underline{F_1} \times (\underline{F_1} + \underline{F_2})|$$

$$= |\underline{F_1} \times \underline{F_1} + \underline{F_1} \times \underline{F_2}|$$

$$= |\underline{0} + \underline{F_1} \times \underline{F_2}| \qquad (\because \underline{F_1} \times \underline{F_2} = \underline{0})$$

$$= |\underline{F_1} \times \underline{F_2}|$$

$$F_1 R \sin\theta = F_1 F_2 \sin\alpha$$

$$R\sin\theta = F_2 \sin\alpha \longrightarrow ③$$

Also

$$\underline{F_1}\cdot\underline{R} = \underline{F_1}\cdot(\underline{F_1}+\underline{F_2})$$

$$= \underline{F_1}\cdot\underline{F_1} + \underline{F_1}\cdot\underline{F_2}$$

$$F_1 R \cos\theta = F_1^2 + F_1 F_2 \cos\alpha$$

$$R\cos\theta = F_1 + F_2\cos\alpha \longrightarrow ④$$

Dividing ③ by ④

$$\frac{\sin\theta}{\cos\theta} = \frac{F_2 \sin\alpha}{F_1 + F_2\cos\alpha}$$

$$\tan\theta = \frac{F_2 \sin\alpha}{F_1 + F_2\cos\alpha}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{F_2\sin\alpha}{F_1+F_2\cos\alpha}\right)} \longrightarrow ⑤$$

which gives the direction of the resultant in terms of magnitudes, of the forces and angle bet. them.

OR

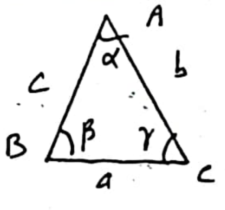
We can also find the direction of the resultant by applying sine rule to triangle O'AC.

$$\frac{R}{\sin(180^\circ-\alpha)} = \frac{F_2}{\sin\theta}$$

$$\Rightarrow \frac{R}{\sin\alpha} = \frac{F_2}{\sin\theta}$$

$$\Rightarrow \boxed{\sin\theta = \frac{F_2\sin\alpha}{R}} \longrightarrow ⑥$$

Revision



Law of Sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

OR

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

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Equation ⑥ gives the direction of the resultant in terms of the magnitudes of the resultant and force F_2 and angle between the forces. This formula can be applied only when the magnitude of the resultant is known.

OR

The formula ⑤ can also be found directly from fig. We have from the fig above

$$AD = F_2 \cos\alpha \qquad CD = F_2 \sin\alpha.$$

$$\begin{aligned} OD &= OA + AD \\ &= F_1 + F_2\cos\alpha \end{aligned}$$

Now from ΔOCD

$$\tan\theta = \frac{CD}{OD} = \frac{F_2 \sin\alpha}{F_1 + F_2\cos\alpha}$$

$$\boxed{\tan\theta = \frac{F_2 \sin\alpha}{F_1 + F_2\cos\alpha}.}$$

Results

1)# Maximum Resultant

We have expression for the resultant

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$

we note that for given forces F_1 & F_2 the magnitude of the resultant depends upon $\cos\alpha$ and is max when $\cos\alpha$ is max i.e

when $\cos\alpha = 1$

or $\alpha = 0$

⇒ The resultant of two concurrent forces is maximum when the forces act in the same direction.

The maximum resultant is found by putting $\alpha = 0^\circ$ or $\cos\alpha = 1$ and is

$$R_{max} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2}$$

$$= \sqrt{(F_1+F_2)^2} = F_1 + F_2$$

$\Rightarrow$ The magnitude of the maximum resultant of two concurrent forces is equal to the sum of the magnitudes of the component forces. The direction of the maximum resultant will be that of the two forces because its line of action coincide with the collinear lines of action of two forces.

2) # Minimum Resultant

For given two forces the magnitude of the resultant will be minimum when $\cos\alpha$ is min i.e.

when $\cos\alpha = -1$

or $\alpha = \pi$

$\Rightarrow$ The resultant of two concurrent forces is minimum when the forces act in opposite direction. Its magnitude is given by

$$R_{min} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2(-1)}$$

$$= \sqrt{(F_1 - F_2)^2}$$

$$R_{min} = |F_1 - F_2|$$

$\Rightarrow$ Magnitude of the minimum resultant is equal to the +ve difference of the magnitudes of the forces and its direction is that of the greater.

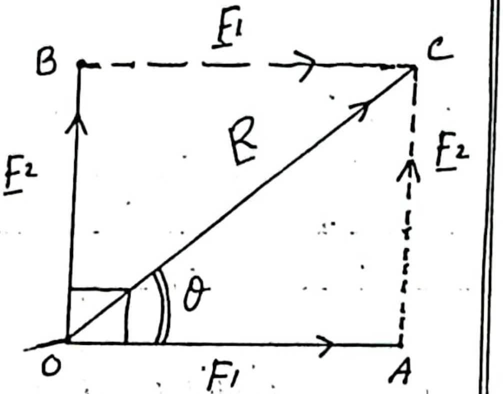
3) # The Resultant of Perpendicular Forces

When forces F_1 and F_2 acting at point O are perpendicular i.e $\alpha = \frac{\pi}{2}$. The magnitude of theire resultant is

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\frac{\pi}{2}}$$

$$= \sqrt{F_1^2 + F_2^2 + 0}$$

$$R^2 = F_1^2 + F_2^2$$



Direction of the resultant is given by

$$\tan\theta = \frac{F_2 \sin\pi/2}{F_1 + F_2\cos\pi/2}$$

$$= \frac{F_2}{F_1 + 0} = \frac{F_2}{F_1}$$

$$\tan\theta = \frac{F_2}{F_1}$$

Here we note some important results as under

(i) If $F_1 = F_2$, then

$\tan\theta = 1 \Rightarrow \theta = \pi/4$

⇒ Resultant bisects the angle between forces.

(ii) If $F_2 > F_1$

Then $\frac{F_2}{F_1} > 1$

$\Rightarrow \tan\theta > 1 \Rightarrow \theta > \frac{\pi}{4}$

⇒ Angle of the resultant is smaller with F_2, force greater in magnitudes.

Physically length of $OB = AC$ will be greater than the length $OA = BC$ and as such line of the action of the resultant is lifted up more i.e its angle with F_1 is greater than the angle with F_2.

4) # The Resultant of Forces of Equal Magnitudes

If the forces F_1 & F_2 have equal magnitudes i.e

$$F_1 = F_2, \text{ then}$$

$$R = \sqrt{F_1^2 + F_1^2 + 2F_1^2 \cos\alpha}$$

$$= \sqrt{2F_1^2 + 2F_1^2 \cos\alpha}$$

$$= \sqrt{2F_1^2(1+\cos\alpha)}$$

$$= \sqrt{2F_1^2 \cdot 2\cos^2(\alpha/2)}$$

$$R = 2F_1 \cos(\alpha/2)$$

$$\boxed{R = 2F_1 \cos \alpha/2}$$

Also the direction of the resultant of the concurrent equal forces is given by

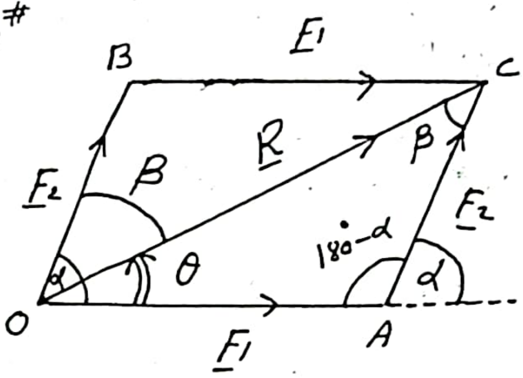
$$\tan\theta = \frac{F_1 \sin\alpha}{F_1 + F_1 \cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$= \frac{2\sin(\alpha/2)\cdot\cos(\alpha/2)}{2\cos^2 \alpha/2} = \tan(\alpha/2)$$

$$\Rightarrow \theta = \alpha/2$$

i.e. The line of action of the resultant bisects the angle between forces

4)# Angles of The Resultant with Component Forces


B
$\underline{F_1}$
C
$\underline{R}$
β
β
$\underline{F_2}$
$\underline{F_2}$
α
θ
180°-α
α
O
$\underline{F_1}$
A

Suppose the resultant of the concurrent forces $\underline{F_1}$, $\underline{F_2}$ makes angles θ, β with $\underline{F_1}$, $\underline{F_2}$ respectively.

By Sines Rule to ΔOAC

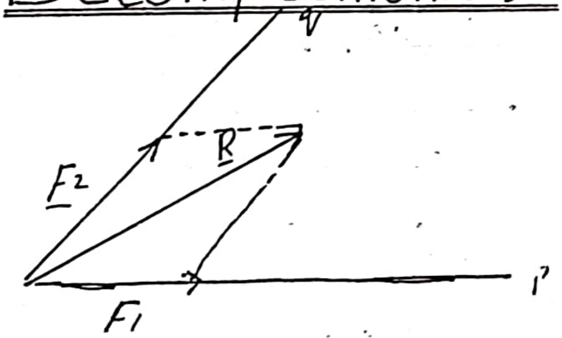
∠BOC = ∠OCA
Alternate angles

$$\frac{R}{\sin(180°-\alpha)} = \frac{F_1}{\sin\beta} = \frac{F_2}{\sin\theta}$$

$$\Rightarrow \frac{R}{\sin\alpha} = \frac{F_1}{\sin\beta} = \frac{F_2}{\sin\theta}$$

$$\Rightarrow \boxed{\begin{aligned} \sin\beta &= \frac{F_1 \sin\alpha}{R} \\ \sin\theta &= \frac{F_2 \sin\alpha}{R} \end{aligned}}$$

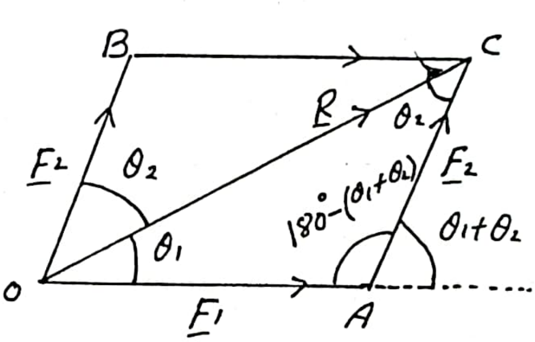
Decomposition of a Force


$\underline{R}$
$\underline{F_2}$
$\underline{F_1}$

The geometric construction of the resultant shows that a given force $\underline{R}$ can be decomposed into an infinite number of pairs of possible components. This can be done by constructing a parallelogram whose diagonal is the given force $\underline{R}$ and whose sides are along any two distinct lines passing through the point of application of $\underline{R}$ and co-planar with the force $\underline{R}$.

Now magnitudes of the components of $\underline{R}$ in any two directions can be found in terms of the magnitude of $\underline{R}$, angles of components with $\underline{R}$ as under

Suppose a force $\underline{R}$ acting at point O (particle at O) is decomposed into two components $\underline{F_1}$ & $\underline{F_2}$ acting along directions of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (i.e two directions at O) respectively making angles θ_1, θ_2 with $\underline{R}$. Then



$$\underline{R} = \underline{F_1} + \underline{F_2} \longrightarrow ①$$

Applying Law of Sines to ΔOAC

$$\frac{R}{\sin(180-(\theta_1+\theta_2))} = \frac{F_1}{\sin\theta_2} = \frac{F_2}{\sin\theta_1}$$

$$\frac{R}{\sin(\theta_1+\theta_2)} = \frac{F_1}{\sin\theta_2} = \frac{F_2}{\sin\theta_1}$$

$$\Rightarrow \boxed{\begin{aligned} F_1 &= \frac{R\sin\theta_2}{\sin(\theta_1+\theta_2)} \\ F_2 &= \frac{R\sin\theta_1}{\sin(\theta_1+\theta_2)} \end{aligned}} \longrightarrow \text{Components of } R \text{ in the direction of } \overrightarrow{OA} \text{ \& } \overrightarrow{OB}$$

The forces $\underline{F_1}$ and $\underline{F_2}$ are sometimes called vector components of $\underline{R}$ and their magnitudes are called scalar Components of $\underline{R}$

OR

We can calculate the above relations as

$$|\underline{F}_1 \times \underline{R}| = |\underline{F}_1 \times (\underline{F}_1 + \underline{F}_2)|$$

$$F_1 R \sin\theta_1 = |\underline{F}_1 \times \underline{F}_2| \quad \because \underline{F}_1 \times \underline{F}_1 = \underline{0}$$

$$F_1 R \sin\theta_1 = F_1 F_2 \sin(\theta_1+\theta_2)$$

$$F_2 = \frac{R\sin\theta_1}{\sin(\theta_1+\theta_2)}$$

and

$$|\underline{F}_2 \times \underline{R}| = |\underline{F}_2 \times (\underline{F}_1 + \underline{F}_2)|$$

$$F_2 R \sin\theta_2 = |\underline{F}_2 \times \underline{F}_1|$$

$$F_2 R \sin\theta_2 = F_1 F_2 \sin(\theta_1+\theta_2)$$

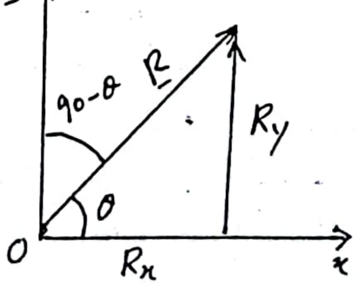
$$F_1 = \frac{R\sin\theta_2}{\sin(\theta_1+\theta_2)}$$

Resolved Parts of Force#

Two Components of a force which act along mutually perpendicular directions are called the resolved parts of the given force along specified directions.

Method to Find The Resolved Part of Force#

Suppose a force $\underline{R}$ makes angle θ with x-axis. Then its Components R_x, R_y along axes are resolved parts from Fig



$$R_x = R\cos\theta \longrightarrow ①$$

$$R_y = R\sin\theta$$

$$= R\sin(\theta)$$

$$R_y = R\cos(90^\circ-\theta) \longrightarrow ②$$

From ① and ② we have a rule to find the resolved part of - in a particular direction:

To find the resolved part of a force in any direction we multiply the magnitude of force with the cosine of the angle between the given force and the given direction.

Also

$$R_x = R\cos\theta = \underline{R}\cdot\hat{i} \longrightarrow ③$$

$$R_y = R\sin\theta = R\cos(90-\theta)$$

$$R_y = \underline{R}\cdot\hat{j} \longrightarrow ④$$

From ③ and ④ we have another rule to find the resolved part in a particular direction

To find the resolved part of a force in a given direction we take dot product of the force with unit vector of that direction.

Algebraic Sum

The sum which is obtained by adding +ve ~~and~~ or -ve terms. The sum $x-y+z$ is an algebraic sum because it is same as $x+(-y)+z$

Arithmetic Sum

The sum of +ve numbers is called arithmetic sum.

Components

Theorem # (Theorem of Resolved Parts)

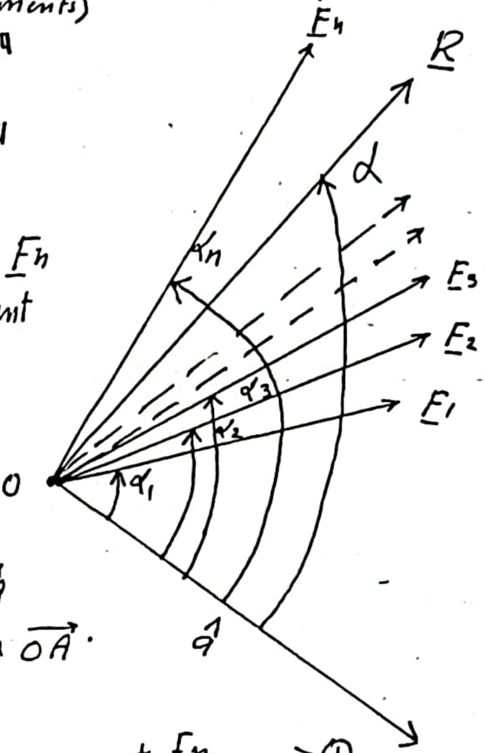
The algebraic sum (sum considering signs i.e +ve, -ve) of the (Components) resolved parts of a system of concurrent forces in any direction is equal to the resolved part (Component) of the resultant of forces in that direction.

OR

The resolved (Compment) part of the resultant of a system of concurrent forces in any direction is equal to the algebraic sum of the resolved parts (Compments) of the forces in that direction

Proof#

Suppose a system of n concurrent forces $\underline{F_1}, \underline{F_2} --- \underline{F_n}$ ($\underline{F_i} : i = 1, 2, ---- n$) acts at point O and $\overrightarrow{OA}$ is any direction at point O. Let $\underline{R}$ be the resultant of the forces. Suppose the forces $\underline{F_1}, \underline{F_2} --- \underline{F_n}$ make angles $\alpha_1, \alpha_2 ---- \alpha_n$ with $\overrightarrow{OA}$ and $\underline{R}$ makes an angle α with $\overrightarrow{OA}$.



Now

$$\underline{R} = \underline{F_1} + \underline{F_2} + \underline{F_3} + ------ + \underline{F_n} \longrightarrow ①$$

Let $\hat{a}$ be the unit vector along OA.
Taking dot product of ① with $\hat{a}$

$$\underline{R} \cdot \hat{a} = \underline{F_1} \cdot \hat{a} + \underline{F_2} \cdot \hat{a} + ----- + \underline{F_n} \cdot \hat{a}$$

$$\underline{R} \cdot \hat{a} = \sum_{i=1}^{n} \underline{F_i} \cdot \hat{a} = \sum_{i=1}^{n} F_i \cdot \cos\alpha_i \longrightarrow ②$$

Since $\underline{F_1} \cdot \hat{a}$ = Component of $\underline{F_1}$ along $\overrightarrow{OA}$
$\underline{F_2} \cdot \hat{a}$ = " " $\underline{F_2}$ " "
-- -- -- -- ---- -- -- -- -- --
$\underline{F_n} \cdot \hat{a}$ = " " $\underline{F_n}$ " "
$\underline{R} \cdot \hat{a}$ = " " $\underline{R}$ " "

Therefore from ② we have
The Component of the Resultant along $\overrightarrow{OA}$ = The algebraic

Sum of the Components of forces $\underline{F}_1, \underline{F}_2, \cdots \underline{F}_n$

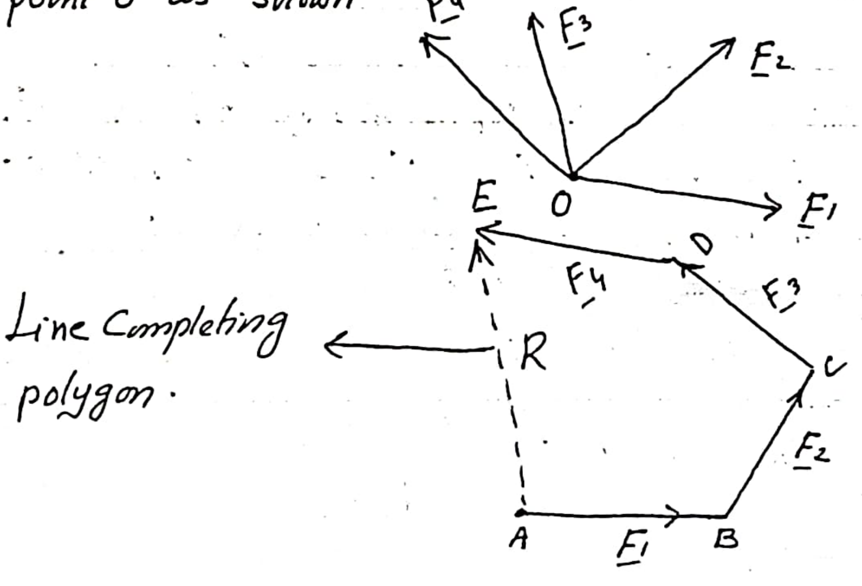
Remarks #(1) Since the components of a force are vectors; therefore directions are attached with them. A component of a force enhance the net effect of all the forces in a direction and reduces the net effect the component actingin opposite sense of that direction. Therefor in above theorem algebraic sum is must

(2) The above theorem provides a basis to find the resultant of a system of Concurrent forces by analytic method (method based on algebraic and calculus method)

The Resultant of a System of Co-planar Concurrent Forces

To find the resultant of any number of Co-planar concurrent forces we apply the polygon rule as stated below.

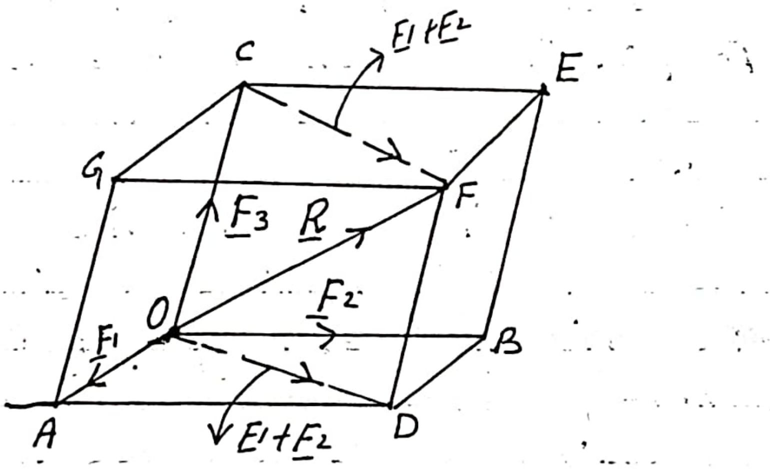
We draw (lines) Consecutive lines representing the given vectors (Forces) in magnitude and direction: the line which completes the polygon represents the resultant e.g Consider four forces $\underline{F}_1, \underline{F}_2, \underline{F}_3, \underline{F}_4$ acting at point O as shown



Remarks # (1) The resultant obtained by the polygon rule is independent of the order of composition of forces because vector addition is associative.

(2) We know that concurrent forces are in equilibrium are in equilibrium if their linear resultant is zero. When the concurrent forces are in equilibrium, the resultant is of zero length i.e given forces themselves can be represented by the sides of a closed polygon i.e length of the line completing the polygon is zero or there is no line completing the polygon and polygon is closed by itself.

The Resultant of Three Non-Coplanar Concurrent Forces (Geometric Method)



Suppose three no-coplanar forces $\underline{F}_1, \underline{F}_2, \underline{F}_3$ acts at point O and are represented by $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$. We complete the parallelopiped with edges $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$. The resultant $\underline{R}$ of forces can be found as.

By parallelogram law

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OD}$$

$$\underline{F}_1 + \underline{F}_2 = \overrightarrow{OD}$$

Again by llgram law

$$\overrightarrow{OF} = \overrightarrow{OD} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

$\underline{R} = \underline{F_1} + \underline{F_2} + \underline{F_3}$

" Hence the resultant of three concurrent non-coplanar forces acting at O, is represented by the diagonal, drawn through O; of parallelopiped with the given forces acting along adjacent edges.

The Polygon of Forces#

If a particle is in equilibrium under the action of several forces, then these forces may be represented in magnitude and direction but not in position by the sides of a closed polygon taken in order

Note# The necessary and sufficient condition for a system of concurrent forces to be in equilibrium is that the polygon of forces is closed i.e their resultant is zero.

If there are only three forces acting on the particle, then we have a triangle of forces. In this case we have

The Triangle of Forces#

If a particle is in equilibrium under the action of three forces, then these forces may be represented in magnitude and direction but not in position by the sides of a triangle taken in order and this triangle is called triangle of forces.

The Resultant of a System of Con-current Forces#

Let the forces $\underline{F_1}, \underline{F_2}, \underline{F_3}, \ldots\ldots, \underline{F_n}$ act at point O and have direction Cosines ($\cos\alpha_i, \cos\beta_i, \cos\gamma_i$) $i = 1, 2, \ldots n$ respectively. Let their resultant R has direction Cosines ($\cos\alpha, \cos\beta, \cos\gamma$). If X, Y, Z are the Components of R along x-axis, y-axis and z-axis.

The by theorem of resolved Components.

$$X = R\cos\alpha = F_1\cos\alpha_1 + F_2\cos\alpha_2 + \cdots + F_n\cos\alpha_n$$

$$= \sum_{i=1}^{n} F_i\cos\alpha_i \longrightarrow ①$$

$$Y = R\cos\beta = F_1\cos\beta_1 + F_2\cos\beta_2 + \cdots + F_n\cos\beta_n$$

$$= \sum_{i=1}^{n} F_i\cos\beta_i \longrightarrow ②$$

$$Z = R\cos\gamma = \sum_{i=1}^{n} F_i\cos\gamma_i \longrightarrow ③$$

Squaring and adding ① . ② & ③

$$\boxed{R = \sqrt{X^2 + Y^2 + Z^2}} \longrightarrow ④$$

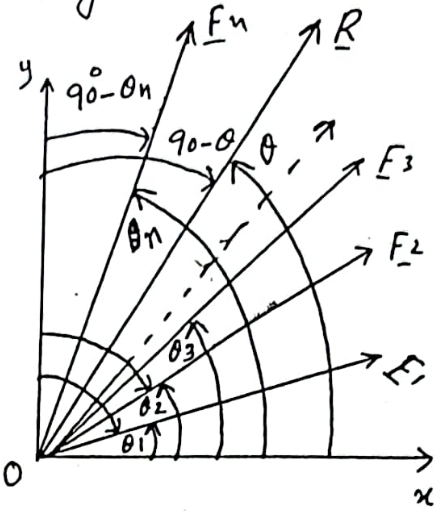
When the magnitude of the resultant has been found from Equation ④, the direction of the resultant can be found by Equations

$$\boxed{\begin{aligned} \cos\alpha &= \frac{X}{R} \\ \cos\beta &= Y/R \\ \cos\gamma &= Z/R \end{aligned}}$$

For Co-planar forces $F_1, F_2 \ldots F_n$ acting at a point O and making angles $\theta_1, \theta_2, \ldots$ on

, the axes ox, oy are taken in the plane of forces.

If the Resultant $\underline{R}$ makes angle θ with x-axis, then we have.



Angles of $\underline{F_1}, \underline{F_2}, \dots \underline{F_n}$ with y-axis are $90^\circ-\theta_1$, $90^\circ-\theta_2$, $90^\circ-\theta_3$, $\dots$ $90^\circ-\theta_n$ respectively and the angle of $\underline{R}$ with y-axis is $90^\circ-\theta$.

By Theorem of Components

$$R\cos\theta = X = F_1\cos\theta_1 + F_2\cos\theta_2 + \dots + F_n\cos\theta_n$$

$$= \sum_{i=1}^{n} F_i\cos\theta_i \longrightarrow ⑤$$

$$R\cos(90^\circ-\theta) = F_1\cos(90^\circ-\theta_1) + F_2\cos(90^\circ-\theta_2) + \dots + F_n\cos(90^\circ-\theta_n)$$

$$Y = R\sin\theta = F_1\sin\theta_1 + F_2\sin\theta_2 + \dots + F_n\sin\theta_n$$

$$= \sum_{i=1}^{n} F_i\sin\theta_i \longrightarrow ⑥$$

Squaring and adding ⑤ & ⑥

$$\boxed{R^2 = \sqrt{X^2+Y^2}} \longrightarrow ⑦$$

and direction of the resultant is given by

$$\boxed{\tan\theta = \frac{Y}{X}}$$

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Another Method to Find The Resultant of System of Concurrent Forces

The resultant found in equations ④ & ⑦ can be expressed in a different form as

We have

$$\underline{R} = \underline{F_1} + \underline{F_2} + \cdots\cdots + F_n$$

$$= \underline{F_1} + \underline{F_2} + \cdots \underline{F_i} + \cdots + F_j + \cdots + \underline{F_n} \quad i<j$$

$$R^2 = \underline{R}\cdot\underline{R} = (\underline{F_1}+\underline{F_2}+\cdots+\underline{F_i}+\cdots+\underline{F_j}+\cdots+\underline{F_n})\cdot(\underline{F_1}+\underline{F_2}+\cdots+\underline{F_i} + \cdots + \underline{F_j}+\cdots+\underline{F_n})$$

$$= F_1^2+F_2^2+F_3^2+\cdots+F_n^2 + \underline{F_1}\cdot\underline{F_2} + \underline{F_1}\cdot F_3 + \cdots \underline{F_1}\cdot\underline{F_n} + \underline{F_2}\cdot\underline{F_1} + \underline{F_2}\cdot\underline{F_3}+\cdots+\underline{F_2}\cdot\underline{F_n} + \underline{F_1}\cdot\underline{F_3}+\underline{F_2}\cdot\underline{F_3}+\underline{F_3}\cdot\underline{F_4} + \cdots + \underline{F_3}\cdot\underline{F_n} + \cdots + \underline{F_1}\cdot F_n + \underline{F_2}\cdot\underline{F_n}+\cdots+\underline{F_{n-1}}\cdot\underline{F_n}$$

$$= \cancel{\sum_{i=1}^{n} F_i^2 + \sum_{j=2}^{n} \underline{F_1}\cdot\underline{F_j} + \sum \underline{F_2}\cdot F}$$

$$= \sum_{i=1}^{n} F_i^2 + 2(\underline{F_1}\cdot\underline{F_2} + \underline{F_1}\cdot\underline{F_3} + \cdots + \underline{F_1}\cdot F_n)$$

$$+ 2(\underline{F_2}\cdot\underline{F_3} + \underline{F_2}\cdot\underline{F_4} + \cdots + \underline{F_2}\cdot\underline{F_n})$$

$$+ 2(\underline{F_3}\cdot\underline{F_4} + \underline{F_3}\cdot\underline{F_5} + \cdots + \underline{F_3}\cdot\underline{F_n})$$

$$- - - - - - - - - - - - -$$

$$+ 2(\underline{F_{n-1}}\cdot\underline{F_n})$$

$$= \sum_{i=1}^{n} F_i^2 + 2\sum_{j=2}^{n} \underline{F_1}\cdot F_j + 2\sum_{j=3}^{n} \underline{F_2}\cdot F_j + \cdots + 2(\underline{F_{n-1}}\cdot\underline{F_n})$$

$$= \sum_{i=1}^{n} F_i^2 + 2\sum_{i<j}\sum \underline{F_i}\cdot\underline{F_j}$$

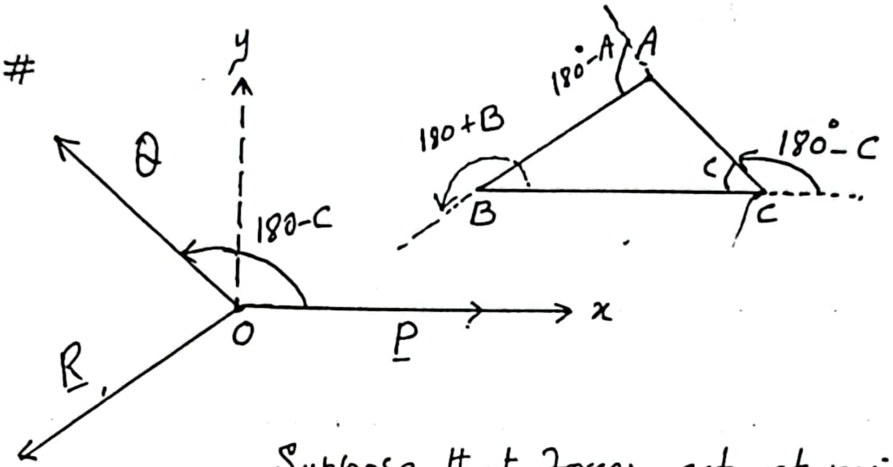
$$\boxed{\begin{aligned} R^2 &= \sum_{i=1}^{n} F_i^2 + 2\sum_{i<j}\sum \underline{F_i}\cdot\underline{F_j} \\ R^2 &= \sum_{i=1}^{n} F_i^2 + 2\sum_{i<j}\sum F_i F_j \cos\theta_{ij} \end{aligned}}$$

where θ_{ij} denotes angle b/w forces $\underline{F_i}$ $\underline{F_j}$

Example # Forces $\underline{P}, \underline{Q}, \underline{R}$ act at a point parallel to the sides of triangle ABC taken in same order. Show that the magnitude of the resultant is

$$(P^2+Q^2+R^2-2QR\cos A-2RP\cos B-2PQ\cos C)^{1/2}$$

Sol #



Suppose that forces act at point O such that $\underline{P}$ is parallel to side BC, $\underline{Q}$ is parallel to CA and $\underline{R}$ is parallel to side AB respectively.

We take ox, oy rectangular axes such that ox is parallel to side BC and $\underline{P}$ is along it.

$\angle POQ = 180° - C$ $\qquad$ $\angle POR = 180° + B$

Let X, Y be the resolved parts of the resultant along x-axis & y-axis. Then

$$X = P\cos 0° + Q\cos(180°-C) + R\cos(180°+B)$$
$$= P - Q\cos C - R\cos B \longrightarrow ①$$

$$Y = P\sin 0° + Q\sin(180°-C) + R\sin(180°+B)$$
$$= 0 + Q\sin C - R\sin B$$
$$= Q\sin C - R\sin B \longrightarrow ②$$

$$X^2 = (P - Q\cos C - R\cos B)^2$$
$$= P^2 + Q^2\cos^2 C + R^2\cos^2 B - 2PQ\cos C - 2PR\cos B$$
$$+ 2QR\cos B\cos C$$

$$Y^2 = (Q\sin C - R\sin B)^2$$

$$Y^2 = Q^2\sin^2 C + R^2\sin^2 B - 2QR\sin B\sin C$$

$$R^2 = X^2 + Y^2$$

$$= P^2 + Q^2(\cos^2 C + \sin^2 C) + R^2(\sin^2 B + \cos^2 B) + 2QR(\cos B\cos C - \sin B\sin C) - 2PQ\cos C - 2PR\cos B$$

$$= P^2 + Q^2 + R^2 + 2RQ\cos(B+C) - 2PQ\cos C - 2PR\cos B$$

$$= P^2 + Q^2 + R^2 + 2RQ\cos(180^\circ - A) - 2PQ\cos C - 2PR\cos B$$

$$= P^2 + Q^2 + R^2 - 2RQ\cos A - 2PQ\cos C - 2PR\cos B$$

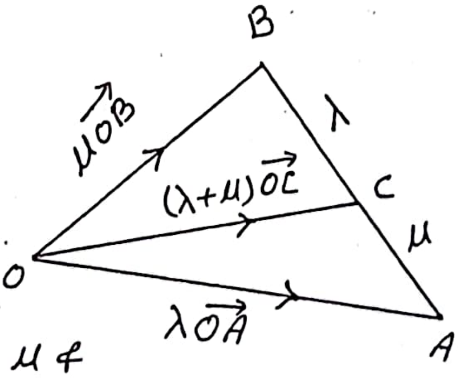
$$\because A+B+C = 180^\circ$$
$$\Rightarrow B+C = 180^\circ - A$$

$$R = (P^2 + Q^2 + R^2 - 2RQ\cos A - 2PQ\cos C - 2PR\cos B)^{1/2}$$

The (λ, μ) Theorem#

Statement# If two concurrent forces are represented by $\lambda\overrightarrow{OA}$ and $\mu\overrightarrow{OB}$ respectively, their resultant is given by $(\lambda+\mu)\overrightarrow{OC}$, where C is a point which divides AB in ratio $\mu : \lambda$ i.e $AC : CB = \mu : \lambda$

Proof# The forces are as shown


B
λ
$\mu\overrightarrow{OB}$
$(\lambda+\mu)\overrightarrow{OC}$
C
μ
O
$\lambda\overrightarrow{OA}$
A

From fig

$$\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA} \longrightarrow (1)$$
$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} \longrightarrow (2)$$

Multiplying (1) by λ & (2) by μ & adding, we get

$$\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}=(\lambda+\mu)\overrightarrow{OC}+\lambda\overrightarrow{CA}+\mu\overrightarrow{CB} \longrightarrow ③$$

$$\because \quad \frac{AC}{CB}=\frac{\mu}{\lambda} \Rightarrow \lambda(AC)=\mu(CB)$$

$$\therefore \quad \mu\overrightarrow{CB}=\lambda\overrightarrow{AC}=-\lambda\overrightarrow{CA}$$

$$\Rightarrow \quad \lambda\overrightarrow{CA}+\mu\overrightarrow{CB}=\underline{0} \longrightarrow ④$$

Using ④ in ③

$$\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}=(\lambda+\mu)\overrightarrow{OC}$$

Corollary # 1

$$\lambda\overrightarrow{OA}+\mu\overrightarrow{OB}=(\lambda+\mu)\overrightarrow{OC}$$

$$\text{or} \quad -\lambda\overrightarrow{AO}-\mu\overrightarrow{BO}=-(\lambda+\mu)\overrightarrow{CO}$$

$$\text{or} \quad \lambda\overrightarrow{AO}+\mu\overrightarrow{BO}=(\lambda+\mu)\overrightarrow{CO}$$

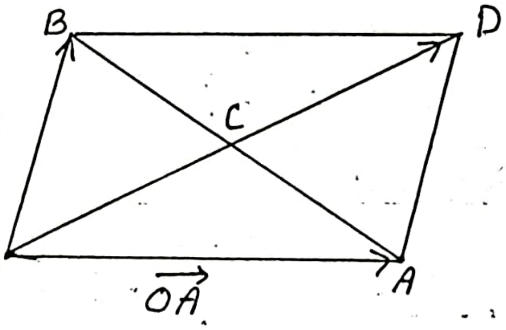
This shows that (λ, μ) theorem remain valid if the forces are directed towards O

Cor # 2. Deduction of the Parallelogram Law

If $\lambda=\mu=1$, so that C is the middle point of AB. Then (λ,μ) Theorem gives

$$\overrightarrow{OA}+\overrightarrow{OB}=2\overrightarrow{OC}=\overrightarrow{OD}$$

where OD is the diagonal of the parallelogram OADB.



Since the diagonals of a parallelogram bisect each other, therefore C is a middle point of OD and so

$$OD=2OC$$

$$\Rightarrow \overrightarrow{OD}=2\overrightarrow{OC}$$

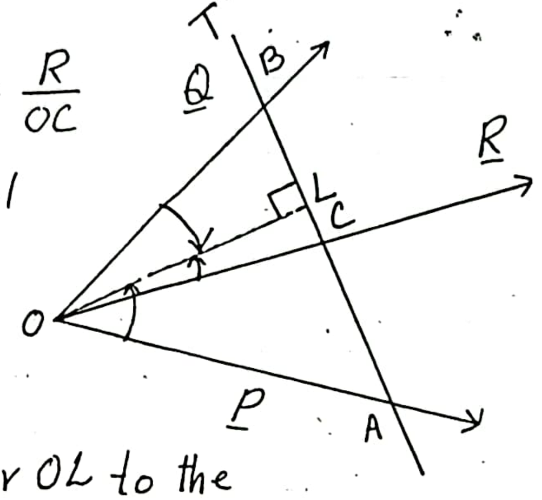
Hence $\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OD}$

which is the parallelogram law of forces.

Remarks # (λ, μ) Theorem is an extension of the parallelogram law of forces.

Example # Forces $\underline{P}, \underline{Q}, \underline{R}$ act at a point O and their resultant is $\underline{R}$. If any transversal cuts the lines of action of the forces $\underline{P}, \underline{Q}, \underline{R}$ at points A, B, C respectively, prove that

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$$



Sol # Let the transversal T cuts the lines of action of forces $\underline{P}, \underline{Q}, \underline{R}$ at points A, B, C respectively.

Draw a perpendicular OL to the transversal line T.

Now the sum of the components of $\underline{P}, \underline{Q}$ along OL is equal to the component of R along OL.

Resolving the forces along OL

$$P\cos\angle AOL + Q\cos\angle BOL = R\cos\angle COL \longrightarrow \textcircled{1}$$

In ΔAOL

$$\frac{OL}{OA} = \cos\angle AOL$$

In ΔBOL $\quad \dfrac{OL}{OB} = \cos\angle BOL$

In ΔCOL $\quad \dfrac{OL}{OC} = \cos\angle COL$

using these in ①

$$P\left(\frac{OL}{OA}\right) + Q\left(\frac{OL}{OB}\right) = R\left(\frac{OL}{OC}\right)$$

$$\Rightarrow \frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$$

Method-II # $\because$ $\underline{P}$ is along $\overrightarrow{OA}$

$\therefore$ Their unit vectors are same and so

$$\underline{P} = P\left(\frac{\overrightarrow{OA}}{OA}\right)$$

Similarly

$$\underline{Q} = Q\left(\frac{\overrightarrow{OB}}{OB}\right) \qquad \underline{R} = R\left(\frac{\overrightarrow{OC}}{OC}\right)$$

Now

$$\underline{R} = \underline{P} + \underline{Q}$$

$$\Rightarrow R\left(\frac{\overrightarrow{OC}}{OC}\right) = P\left(\frac{\overrightarrow{OA}}{OA}\right) + Q\left(\frac{\overrightarrow{OB}}{OB}\right)$$

$$\Rightarrow \left(\frac{R}{OC}\right)\overrightarrow{OC} = \left(\frac{P}{OA}\right)\overrightarrow{OA} + \left(\frac{Q}{OB}\right)\overrightarrow{OB}$$

$$= \left(\frac{P}{OA} + \frac{Q}{OB}\right)\overrightarrow{OC}$$

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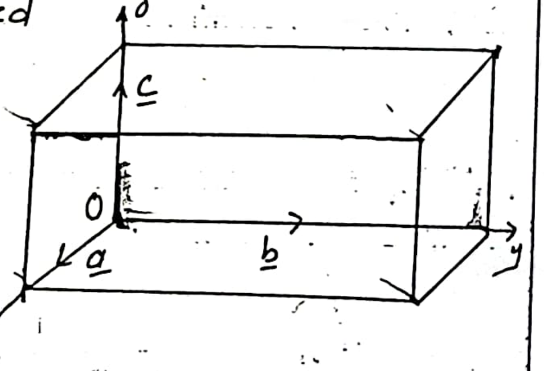
By (λ, μ) Theorem on R.H.S with $AC : CB = \frac{Q}{OB} : \frac{P}{OA}$

$$\Rightarrow \frac{R}{OC} = \frac{P}{OA} + \frac{Q}{OB}$$

Scalar Triple Product of Vectors

The scalar triple of three vectors $\underline{a}, \underline{b}, \underline{c}$ is given by $\underline{a} \cdot \underline{b} \times \underline{c} = [\underline{a}\ \underline{b}\ \underline{c}]$.

Geometrically if $\underline{a}, \underline{b}, \underline{c}$ are three adjacent edges of a parallelopiped, then $\underline{a} \cdot \underline{b} \times \underline{c}$ gives the Volume of the parallelopiped



Note # Three vectors $\underline{a}, \underline{b}$ $\underline{c}$ are Co-planar if volume of the parallelopiped with Co-terminous edges $\underline{a}, \underline{b}, \underline{c}$ is zero i.e if

$$\underline{a} \cdot \underline{b} \times \underline{c} = [\underline{a}\ \underline{b}\ \underline{c}] = 0$$

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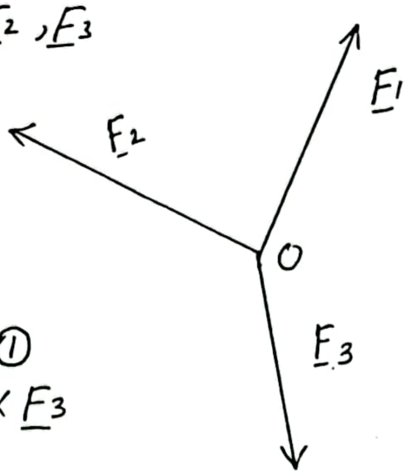
Theorem # If three forces acting at a particle (i.e concurrent) are in equilibrium, then the forces must be co-planar.

Proof # Let the forces $\underline{F}_1, \underline{F}_2, \underline{F}_3$ act at a particle at point O.

Since the forces are in equilibrium, therefore



$$\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = \underline{0} \longrightarrow ①$$

Multiplying scalarly by $\underline{F}_2 \times \underline{F}_3$

$$\underline{F}_1 \cdot \underline{F}_2 \times \underline{F}_3 = \underline{F}_2 \cdot \underline{F}_2 \times \underline{F}_3 + \underline{F}_3 \cdot \underline{F}_2 \times \underline{F}_3$$

$$\Rightarrow \underline{F}_1 \cdot F_2 \times \underline{F}_3 = 0 + 0$$

$$\Rightarrow \underline{F}_1 \cdot \underline{F}_2 \times \underline{F}_3 = 0$$

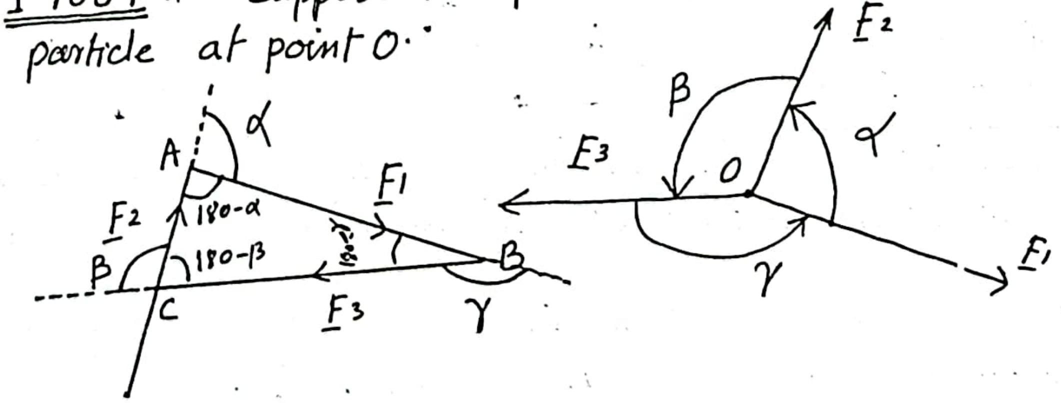
$\Rightarrow \underline{F}_1, \underline{F}_2, \underline{F}_3$ are co-planar

---

Lami's Theorem

Statement # If three forces acting on a particle (i.e concurrent) are in equilibrium, then the magnitude of each force is proportional to the sine of the angle between the other two.

Proof # Suppose the forces $\underline{F}_1, \underline{F}_2, \underline{F}_3$ act at a particle at point O.



Since the forces are in equilibrium, therefore these can be represented in magnitude and position along the sides of triangle ABC shown.

Let the angle b/w $\underline{F_1}, \underline{F_2} = \alpha$

" " " $F_2, F_3 = \beta$

" " " $F_3, F_1 = \gamma$.

Then α, β, γ are exterior angles of triangle of forces ABC.

Applying the Law of sines to the triangle of forces ABC

$$F_1 : F_2 : F_3 = AB : AC : BC$$

$$= \sin\angle ACB : \sin\angle ABC : \sin\angle BAC$$

$$= \sin(180^\circ-\beta) : \sin(180^\circ-\gamma) : \sin(180^\circ-\alpha)$$

$$= \sin\beta : \sin\gamma : \sin\alpha$$

or

$$\frac{F_1}{\sin\beta} = \frac{F_2}{\sin\gamma} = \frac{F_3}{\sin\alpha}. \quad \text{Proved.}$$

OR

Second Method# Since the forces are in equilibrium, therefore we have

$$\underline{F_1} + \underline{F_2} + \underline{F_3} = \underline{0} \quad \rightarrow ①$$

Taking Cross-product of ① with $\underline{F_1}$

$$\underline{F_1} \times \underline{F_1} + \underline{F_1} \times \underline{F_2} + \underline{F_1} \times \underline{F_3} = \underline{0}$$

$$\underline{0} + \underline{F_1} \times \underline{F_2} + \underline{F_1} \times \underline{F_3} = \underline{0}$$

$$\underline{F_1} \times \underline{F_2} = \underline{F_3} \times \underline{F_1} \quad \rightarrow ②$$

Taking Cross-product of ① with $\underline{F_2}$ we have

$$\underline{F_1} \times \underline{F_2} + \underline{0} + \underline{F_3} \times \underline{F_2} = \underline{0}$$

$$\Rightarrow \underline{F_1} \times \underline{F_2} = \underline{F_2} \times \underline{F_3} \quad \rightarrow ③$$

By ② & ③

$$\underline{F_1} \times \underline{F_2} = \underline{F_2} \times \underline{F_3} = \underline{F_3} \times \underline{F_1},$$

$$\Rightarrow |\underline{F_1} \times \underline{F_2}| = |\underline{F_2} \times \underline{F_3}| = |\underline{F_3} \times \underline{F_1}|$$

$$\Rightarrow F_1 F_2 \sin\alpha = F_2 F_3 \sin\beta = F_1 F_3 \sin\gamma$$

Dividing by $F_1 F_2 F_3$

$$\frac{\sin\alpha}{F_3} = \frac{\sin\beta}{F_1} = \frac{\sin\gamma}{F_2}$$

Reciprocating.

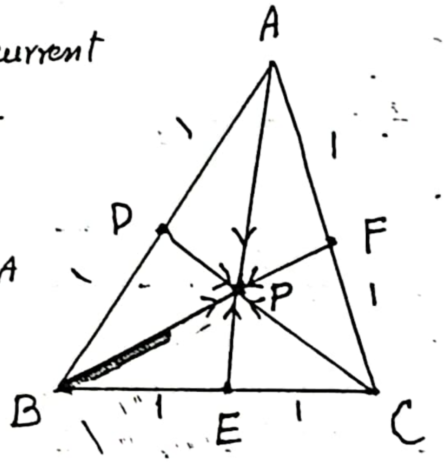
$$\frac{F_1}{\sin\beta} = \frac{F_2}{\sin\gamma} = \frac{F_3}{\sin\alpha}$$

$\Rightarrow F_1 : F_2 : F_3 = \sin\beta : \sin\gamma : \sin\alpha$

Example # P is any point in the plane of a Triangle ABC and D, E, F are middle points of its sides, prove that forces $\overrightarrow{AP}$, $\overrightarrow{BP}$, $\overrightarrow{CP}$, $\overrightarrow{PD}$, $\overrightarrow{PE}$, $\overrightarrow{PF}$ are in equilibrium.

Sol # Since force are concurrent at point P, therefore these will be in equilibrium if their resultant is zero

By (λ, μ) Theorem from ΔBPA



$$1 \cdot \overrightarrow{BP} + 1 \cdot \overrightarrow{AP} = (1+1)\overrightarrow{DP}$$

$$\overrightarrow{BP} + \overrightarrow{AP} = 2\overrightarrow{DP} \longrightarrow ①$$

Similarly

$$\overrightarrow{BP} + \overrightarrow{CP} = 2\overrightarrow{EP} \longrightarrow ②$$

$$\overrightarrow{CP} + \overrightarrow{AP} = 2\overrightarrow{FP} \longrightarrow ③$$

$$\overrightarrow{AP} + \overrightarrow{BP} = 2\overrightarrow{DP} \text{ (repeated)}$$

Adding ① ② & ③

$$2\overrightarrow{AP} + 2\overrightarrow{BP} + 2\overrightarrow{CP} = 2(\overrightarrow{EP} + \overrightarrow{FP} + \overrightarrow{DP})$$

$$\overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} = \overrightarrow{EP} + \overrightarrow{FP} + \overrightarrow{DP}$$

$$\overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} - \overrightarrow{EP} - \overrightarrow{FP} - \overrightarrow{DP} = \underline{0}$$

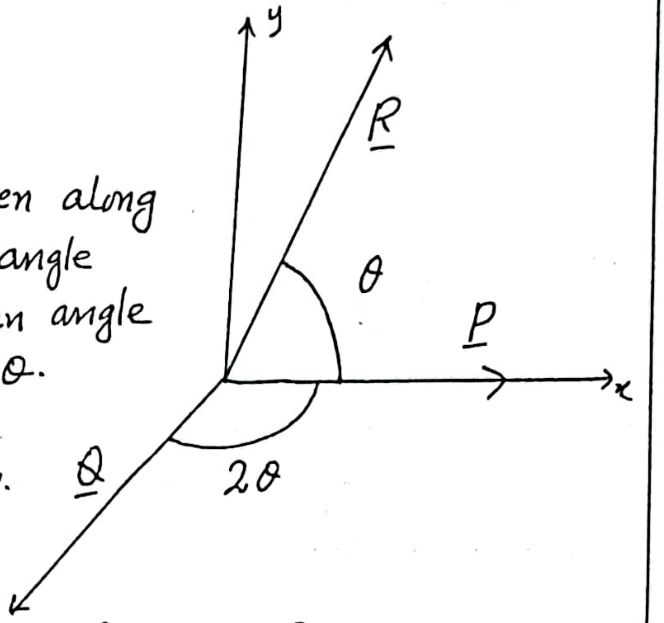
$$\overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} + \overrightarrow{PE} + \overrightarrow{PF} + \overrightarrow{PD} = \underline{0}$$

$\Rightarrow$ forces are in equilibrium

Example # Three forces $\underline{P}, \underline{Q}, \underline{R}$ acting at a point are in equilibrium and the angle between $\underline{P}, \underline{Q}$ is double of the angle between $\underline{P}$ and $\underline{R}$. Prove that

$$R^2 = Q(Q-P)$$

Sol #



Let $\underline{P}$ be taken along x-axis. If θ is the angle between $\underline{P}$ and $\underline{R}$, then angle between $\underline{Q}$ & $\underline{P}$ is 2θ.

Resolving the forces horizontally and vertically.

$$X = P\cos 0^\circ + Q\cos(360^\circ - 2\theta) + R\cos\theta$$
$$= P + Q\cos 2\theta + R\cos\theta \longrightarrow ①$$

$$Y = P\sin 0^\circ + Q\sin(360^\circ - 2\theta) + R\sin\theta$$
$$= 0 - Q\sin 2\theta + R\sin\theta$$

$\because$ Forces are in equilibrium

$\therefore \quad X = 0 \qquad Y = 0$

$$\Rightarrow P + Q\cos 2\theta + R\cos\theta = 0 \longrightarrow ②$$
$$-Q\sin 2\theta + R\sin\theta = 0 \longrightarrow ③$$

from ③

$$R\sin\theta - Q\sin 2\theta = 0$$
$$R\sin\theta - 2Q\sin\theta\cos\theta = 0$$
$$\sin\theta(R - 2Q\cos\theta) = 0$$
$$\Rightarrow R - 2Q\cos\theta = 0 \quad (\because \sin\theta \neq 0 \text{ because } \theta \neq 0)$$
$$\Rightarrow \cos\theta = \frac{R}{2Q} \longrightarrow ④$$

from ②

$$P + Q(2\cos^2\theta - 1) + R\cos\theta = 0$$

putting $\cos\theta = \frac{R}{2Q}$ in it

$$P+Q\left(2\frac{R^2}{4Q^2}-1\right)+R\cdot\frac{R}{2Q}=0$$

$$P+Q\left(\frac{R^2}{2Q^2}-1\right)+\frac{R^2}{2Q}=0$$

$$P+\frac{R^2}{2Q}-Q+\frac{R^2}{2Q}=0$$

$$P-Q+\frac{R^2}{Q}=0$$

$$\Rightarrow\quad \frac{R^2}{Q}=Q-P$$

$$\Rightarrow\quad R^2=Q(Q-P)$$

Second Method

Since the forces are in equilibrium, by Lamis Theorem

$$\frac{P}{\sin(2\pi-3\theta)}=\frac{Q}{\sin\theta}=\frac{R}{\sin2\theta}$$

$$-\frac{P}{\sin3\theta}=\frac{Q}{\sin\theta}=\frac{R}{\sin2\theta}\quad *$$

$$\Rightarrow\quad P\sin\theta+Q\sin3\theta=0\quad\longrightarrow ①$$

$$Q\sin2\theta-R\sin\theta=0\quad\longrightarrow ②$$

from ②

$$\cos\theta=\frac{R}{2Q}\quad\longrightarrow ③$$

from ①

$$P\sin\theta+Q(3\sin\theta-4\sin^3\theta)=0$$

$$P+Q(3-4\sin^2\theta)=0\quad\because \sin\theta\neq0$$

$$P+Q[3-4(1-\cos^2\theta)]=0$$

$$P+Q(4\cos^2\theta-1)=0$$

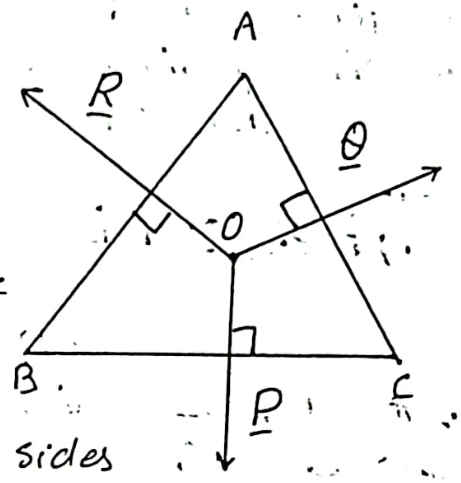
putting $\cos\theta=R/2Q$

$$P+Q\left(\frac{R^2}{Q^2}-1\right)=0$$

$$\Rightarrow\quad R^2=Q(Q-P)\quad\text{Proved.}$$

Example # : Three forces act perpendicularly to the sides of a triangle at their points and are proportional to the sides. Prove that they are in equilibrium

Sol # Let ABC be the triangle and forces P, Q, R act along the perpendicular bisectors of the sides. Suppose that the lines of action of forces intersect at point O.



∵ Forces are proportional to sides of triangle

$$\therefore \quad \frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \quad \rightarrow ①$$

By Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \rightarrow ②$$

By ① & ②

$$\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C} \quad \rightarrow ③$$

From Fig. $\angle QOR = 360 - (180^\circ + A) = 180^\circ - A$

$$\angle ROP = 180^\circ - B$$

$$\angle POQ = 180^\circ - C$$

$$\Rightarrow \sin\angle QOR = \sin(180^\circ - A) = \sin A$$

$$\sin\angle ROP = \sin B$$

$$\sin\angle POQ = \sin C$$

using these in ③

$$\frac{P}{\sin\angle QOR} = \frac{Q}{\sin\angle ROP} = \frac{R}{\sin\angle POQ}$$

which is Lami's Theorem

⇒ forces are in equilibrium

Reference Point of a Rigid Body#

When a body rotates about a fixed point (particle) of it, the fixed point becomes a Reference point for all other particles of the body. The rotation will be observed relative to the fixed point

Types of Displacements Which a Force May Produce

1) # A force acting on particle tends to produce linear displacement in its direction.

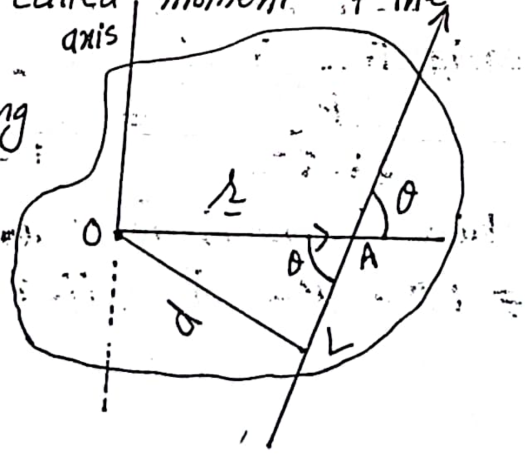
2) # A force acting on a rigid body tends to produce two types of displacements. These are either

(a) Translation # which is rectillinear motion of the body such that each particle of the body undergoes same displacement OR

(b) Rotation # which is about some fixed point in the body which is different for each particle, depending on its position relative to the fixed point

Moment of a Force About a Point

The measure of the turning effect produced by the force is called moment of the force.



Consider a force F acting on a rigid body which tends to rotate the body about point O. Let A be any point on the line of action of force $\underline{F}$. Let $\underline{r} = \overrightarrow{OA}$ be the position vector of the point A w.r.t the fixed reference point O. Then the vector moment $\underline{M}$ of $\underline{F}$ about point O (about an axis through O) is defined as

$$\underline{M} = \underset{O}{\mathrm{Mom}}\,\underline{F} = \overrightarrow{OA} \times \overrightarrow{F}$$
$$= \underline{r} \times \underline{F} \quad \longrightarrow ①$$

If θ is the angle between $\underline{r}$ and $\underline{F}$, then magnitude of the moment of $\underline{F}$ about O is :

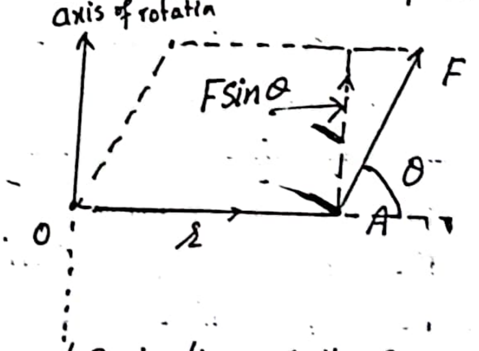
$$|\underline{M}| = |\underline{r} \times \underline{F}| = rF\sin\theta$$
$$= (r\sin\theta)F$$

If $\overline{OL} = d$ is the perpendicular distance from O to the line a action of $\underline{F}$, then $d = r\sin\theta$ and we have

$$|\underline{M}| = \pm dF \quad \longrightarrow ②$$

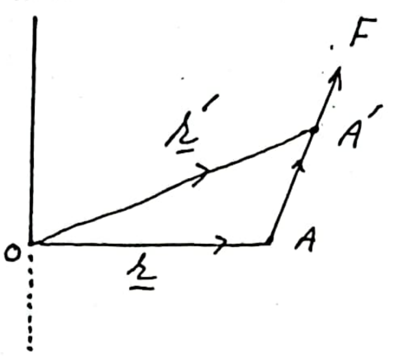
The scalar moment of $\underline{F}$ about O is +ve or −ve according as the rotation is anti-clockwise or clockwise respectively.

Again we note that $F\sin\theta$ is the projection of $\underline{F}$ on a plane perpendicular to the axis of rotation and $|\underline{r}| = r$ is the perpendicular distance of for $F\sin\theta$ (projection) from point O (i.e from axis of rotation). Thus



Moment of a force about an axis = (Projection of the force on a plane ⊥ar to the axis) × (⊥ar distance of the projection from O) $\longrightarrow ③$

By definition of vector product, the moment $\underline{M}$ is perpendicular to the plane containing O and $\underline{F}$ i.e along axis of rotation and its sense is determined by righ-hand rule.



We also note that moment $\underline{M}$ does not depend on the choice of particular point A on the line of action of $\underline{F}$ (i.e is independent of the choice of point A on the line of action of the force $\underline{F}$) which can be proved as

Let A′ be any other point on the line of action of $\underline{F}$ and $\underline{r}'$ be its p.v relative to point O.

Then

$$\underline{r}' = \underline{r} + \overrightarrow{AA'}$$

Now the moment of $\underline{F}$ about O is

$$\underline{M}' = \underline{r}' \times \underline{F}$$

$$= (\underline{r} + \overrightarrow{AA'}) \times \underline{F}$$

$$\underline{r} \times \underline{F} + \overrightarrow{AA'} \times \vec{F}$$

$$= \underline{r} \times \underline{F} + \underline{0} \qquad (\because \vec{F} \text{ is parallel to } \overrightarrow{AA'})$$

$$= \underline{r} \times \underline{F}$$

$$= \underline{M}$$ which is same as before.

We also observe that moment $\underline{M}$ of the force $\underline{F}$ depends only upon

(i) The magnitude of the moment i.e Fd or magnitue of the force and moment arm d,

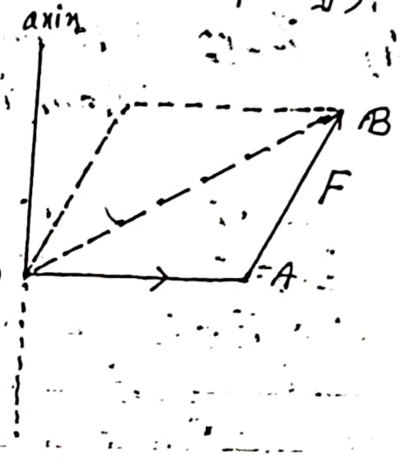
(ii) the plane of rotation through Through the line of action of force and point O

(iii) the sense of rotation in the plane of rotation.

When all the forces acting upon the body are co-planar and O lies in the plane of forces, then the plane of rotation is the plane of forces itself.

Importan Results About Moment

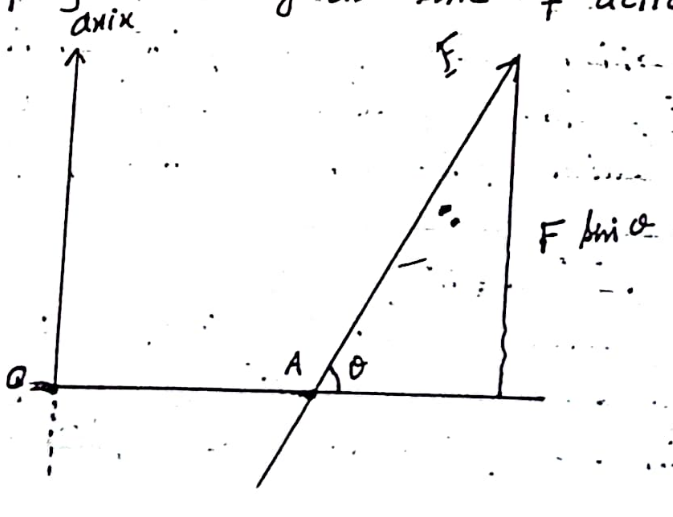
1) If a Force $\underline{F}$ acting on body is represented in magnitude, direction and position (i.e completely) by a line $\overrightarrow{AB}$, then the magnitude of moment of $\underline{F}$ about an axis through point O is given by



$$\text{Moment} = |\overrightarrow{OA} \times \overrightarrow{AB}|$$
$$= \text{Area of the llgram Constructed at } \overrightarrow{OA} \text{ and } \overrightarrow{AB}$$
$$= 2(\text{Area of } \Delta OAB).$$

Thus the moment of force is numerically equal to the area of the parallelogram constructed at $\overrightarrow{OA}$ & $\overrightarrow{OB}$ or is equal to (the) twice the area of ΔOAB.

2)# Since the moment of a force about a given axis of rotation is independent of choice point on the line of action (i.e $\underline{r} \times \underline{F}$ remain same in direction and magnitude if $\underline{r}$ is p.v of any point (not fixed) on the line of action of force), therefore the moment of $\underline{F}$ about the given axis remain same if $\underline{F}$ is made to slide along its line of action without changing magnitude and sense. It can be seen that this only slides the projection along its line of action as



3) The moment about an axis of a force $\underline{F}$ is unaltered if F is moved without change of magnitude, direction in a direction parallel to the axis (This does not change its projection on a plane perpendicular to the axis.

4) Zero Moment #

$$Mom_o \underline{F} = \underline{r} \times \underline{F}$$

If $Mom_o \underline{F} = \underline{r} \times \underline{F} = \underline{0}$, then $\underline{r} = \underline{0}$ or $\underline{F} = \underline{0}$, or $\underline{r}$ and $\underline{F}$ are parallel. Now $\underline{F} \neq \underline{0}$ because we take some non-zero force and $\underline{r}$, $\underline{F}$ can not be parallel. Thus moment of a force about a point is zero when $\underline{r} = 0$ i.e the point (about which moment is taken) lies on the line of action of Force.

Remarks # (1) The term "moment of a force about a point" is meaningless in itself because forces do not cause turning about a point but about a line. The moment about a point is a common abbreviation for the turning effect of a force about an axis passing through the point and which is perpendicular to the plane containing the force and the point.

(2) Given the moment of $\underline{F}$ about O is '$\underline{a}$' we have $\underline{r} \times \underline{F} = \underline{a}$. As $\underline{r}$ is the position vector of any point on the line of action of $\underline{F}$, the equation $\underline{r} \times \underline{F} = \underline{a}$ is the vector equation of the line of action of force $\underline{F}$.

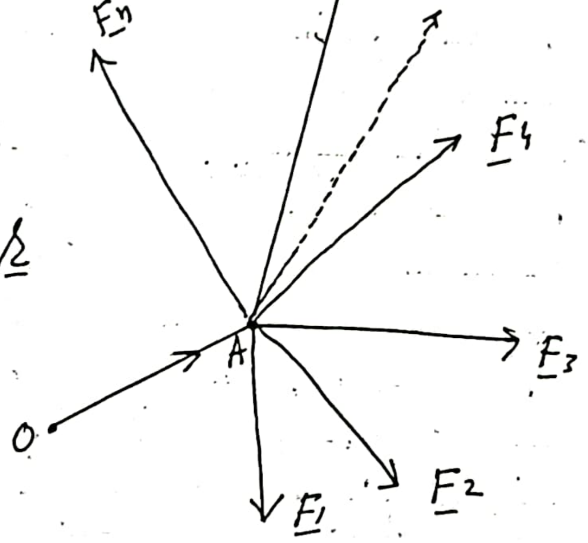
Torque #

One of the ways (of ro) in which rotation can be caused is to apply a force at some distance from the axis or pivot. But there are other means of producing rotation. To turn a door knob, for instance, we simply take hold of it and twist it. The general name given to any turning effect is torque.

Varignon's Theorem

The Sum of moments of a system of Co-planar Concurrent Forces about a point is equal to the moment of the resultant of Forces about that point

Proof



Suppose forces $\underline{F_1}, \underline{F_2}, \underline{F_3}$ ------- $\underline{F_n}$ act at point A and $\underline{r}$ is the position vector of A w.r.t O. If $\underline{R}$ is the resultant of forces, then $\underline{R}$ also passes through O and.

$$\underline{R} = \underline{F_1} + \underline{F_2} + \underline{F_3} + \cdots + \underline{F_n} \longrightarrow ①$$

Multiplying ① vectorially by $\underline{r}$, we have.

$$\underline{r} \times \underline{R} = \underline{r} \times (\underline{F_1} + \underline{F_2} + \underline{F_2} + \cdots + \underline{F_n})$$

$$= \underline{r} \times \underline{F_1} + \underline{r} \times \underline{F_2} + \cdots + \underline{r} \times \underline{F_n}$$

$$\Rightarrow \text{Mom}_O \underline{R} = \text{Mom}_O \underline{F_1} + \text{Mom}_O \underline{F_2} + \cdots + \text{Mom}_O \underline{F_n}$$

which proves the theorem.

Results #(1) The vector sum of the moments of forces about O $= \underline{0}$

$\Rightarrow \quad \text{Mom}_O \underline{R} = \underline{0}$

$\Rightarrow \quad \underline{r} \times \underline{R} = \underline{0}$

$\Rightarrow$ either $\underline{r} = \overrightarrow{OA} = \underline{0}$ or $\underline{R} = \underline{0}$ $(\because \overrightarrow{OA}$ not ||al to $\underline{R})$

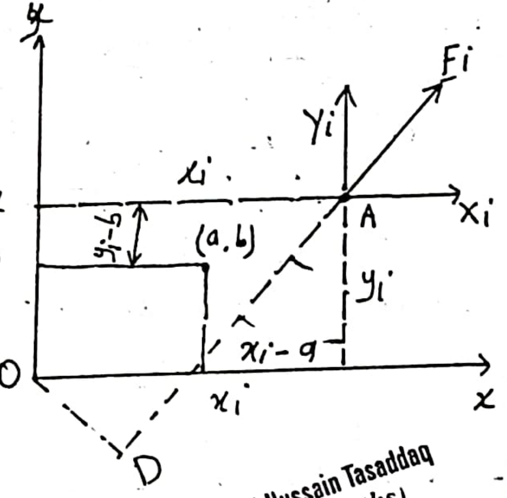
$\Rightarrow$ either pt O lies on the line of action of the resultant or the forces are in equilibrium (Concurrent forces are in equilibrium if their resultant is zero)

Thus in the case of a coplanar system of concurrent forces the sum of the moments (with appropriate signs) of forces about a point O is zero if

(i) The point O lies on the line of action of the resultant

or (ii) the resultant of forces is zero in which case the system is ~~zero~~ in equilibrium (because the system has zero resultant and zero turning effect)

2)) The calculation of the moment of a force $\underline{F_i}$ about point O can be simplified if we resolve the force $\underline{F_i}$ into rectangular components X_i, Y_i parallel to x-axis and y-axis through point O. If x_i, y_i are co-ordinates of a point A on the line of action force $\underline{F_i}$, then by Varignon's theorem the magnitude M_i of the moment of $\underline{F_i}$ about O is given by



M_i = Sum of the magnitudes of moments of X_i and Y_i about O

$= x_i Y_i - y_i X_i$

The moment of $\underline{F_i}$ about point (a, b) is

$M_i = (x_i - a) Y_i - (y_i - b) X_i$

Equipolent Systems of Forces

Two systems of forces are said to be equipolent when following conditions are satisfied.

(i) The vector sum of all the forces of one system is equal to the vector sum of all the forces of the other system i.e. the vector resultants of two forces systems are equipolent.

(ii) The sum of moments of all the forces of one system about an arbitrary line (point) is equal to the sum of the moments of all the forces of the other system about the same line. i.e the turning effects of both systems about the same arbitrary point

Note # The word equivalent is often used. But there is a danger of confusion in using a common word in a technical sense. We might think that the effects of two such force systems were the same; this is not always the case. If we pull a string with equal and opposite forces at its ends, we produce a very different effect from that caused by pushing it with these forces reversed in direction: yet the two systems are equipolent.

Plane Equivallence

Two systems of forces are said to be plane-equivalent for the fundamental plane if

(i) The vector sum of the projections on the fundamental plane of all the forces of one system is equal to the vector sum of the projections on that plane of all forces of the other system.

(vi) The sum of moments of all the forces of one system about an arbitrary line perpendicular to the fundamental plane is equal to the sum of moments of all the forces of the other system about the same line.

Couple

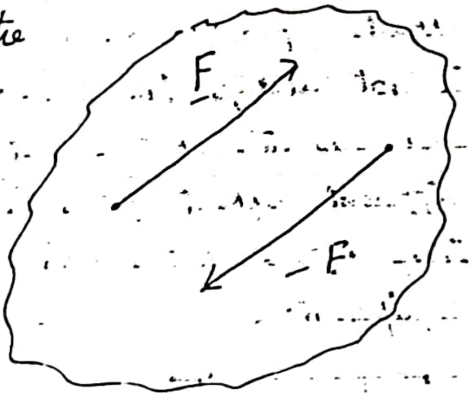
A pair of forces acting on parallel lines, equal in magnitudes and opposite in sense i.e direction, is called a Couple.

The distance between the lines of action of forces is called Couple arm.

OR

Two non-collinear, parallel forces of equal magnitudes and opposite direction acting on a body form a Couple.

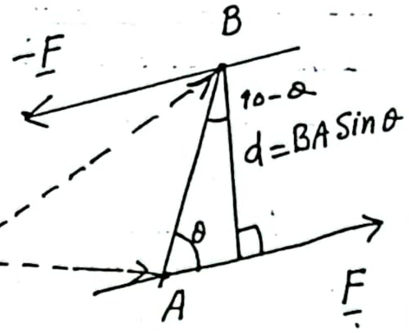
A couple of forces $\underline{F}, -\underline{F}$ is denoted by $(\underline{F}, -\underline{F})$. The vector sum of the forces of a couple is is zero i.e forces of a couple has no resultant. Thus the system of forces of a couple can not replaced by a single force or can not be balanced by a single force. Also forces of couple are not in equilibrium because the forces do not act along the same line.



Moment of Couple

Suppose a Couple $(\underline{F}, -\underline{F})$ acts on a rigid body. Let A, B be any points on the lines of action of forces.

Let P be any point in the plane of couple, Then moment of Couple is



$$G = \text{Mom}_P\, \underline{F} + \text{Mom}_P(-\underline{F})$$

$$= \overrightarrow{PA} \times \overrightarrow{F} + \overrightarrow{PB} \times (-\underline{F})$$

$$= \overrightarrow{PA} \times \overrightarrow{F} - \overrightarrow{PB} \times \overrightarrow{F}$$

$$= (\overrightarrow{PA} - \overrightarrow{PB}) \times \overrightarrow{F}$$

$$= \overrightarrow{BA} \times \overrightarrow{F} \quad \rightarrow ① \qquad \left[\begin{array}{l} \text{as } \overrightarrow{PB} + \overrightarrow{BA} = \overrightarrow{PA} \\ \overrightarrow{BA} = \overrightarrow{PA} - \overrightarrow{PB} \end{array} \right\}$$

$$= \underline{\ell} \times \underline{F} \qquad \overrightarrow{BA} = \underline{\ell}$$

Since $\overrightarrow{BA} \times \overrightarrow{F}$ does not involve position vector of P or p, it means that the sum of moments of the forces forming a couple, about any point in the plane of the couple is constant.

From ①

$$\underline{G} = \underline{\overrightarrow{BA}} \times \underline{F} = \text{Mom}_B\, F$$

Again

$$G = \overrightarrow{BA} \times \overrightarrow{F}$$

$$= -\overrightarrow{AB} \times \overrightarrow{F}$$

$$= \overrightarrow{AB} \times (-\overrightarrow{F})$$

$$= \text{Mom}_A(-\overrightarrow{F})$$

From ① & ② we conclude that the moment of a couple is equal to the moment of the one of the forces of the couple about a point on the line of action of other force.

The magnitude of the couple is

$$|\underline{G}| = |\overrightarrow{BA} \times \overrightarrow{F}|$$

$$= \overline{BA}\, F \sin\theta = Fd$$

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∴ Scalar Moment of the Couple $(\underline{F}, -\underline{F}) = \pm Fd$
where d is perpendicular distance between the lines of action forces and is called length of couple arm. The moment is +ve if the couple tends to rotate a body counter-clockwise and −ve if clockwise.

In two dimensional case the moment of a couple is determined by the product of the magnitude of one of the forces of the couple and the length of its arm with appropriate sign.

Remarks:# (1) $G = \overrightarrow{BA} \times \vec{F}$

∵ Since $\overrightarrow{BA}$ & $\vec{F}$ can never be parallel, therefore moment of a couple is never zero. It therefore follows that a couple acting on a rigid body always tends to rotate it or the turning effect of a couple is never zero.

(2) The plane through the lines of action of the forces of a couple is called plane of the couple. When we have several couples acting in the same plane, there is no need to specify the plane of couple.

(3) Since the vector sum of forces of a couple is zero, therefore two couples in the fundamental plane are plane-equipollent if they have same moment.

(4) Two couples of moments G_1, G_2 in the fundamental plane are together plane-equipolent to a single couple of moment $G_1 + G_2$

(5) The forces of a couple are not in equilibrium because their resultant is zero but their moment about any point is not zero which is 2nd condition for equilibrium.

(6) The moment of a couple depends upon.

(i) the numerical value of moment

(ii) the position of the plane of a couple.

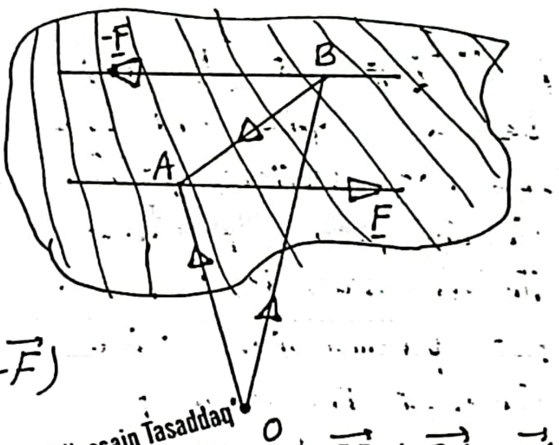
(iii) the sense of rotation in that plane.

(7)# Since the forces of a couple has no resultant, therefore a couple can not be replaced or balanced by a single force.

Moment of a couple Independent of Point not in the Plane of Couple.#

We knew that the moment of a couple is same about any pt. in the plane of the couple. Now consider the sum of moments of the forces of the couple $(F,-F)$ about a point O, not in the plane of the couple.

If A is a point on the line action of F and B is a point on the line of action of $-F$, the sum of the moments of the forces about O is



$$\underline{G} = \vec{OA}\times\vec{F} + \vec{OB}\times(-\vec{F})$$
$$= (\vec{OA} - \vec{OB})\times\vec{F} \qquad \because \vec{OB}+\vec{BA}=\vec{OA}$$
$$= \vec{BA}\times\vec{F}$$

which is independent of the point O

Therefore the vector moment of a couple is the same about any point, whether or not that point is in the plane of the couple.

Equivalent Couples#

Two couples are equivalent

if their vector moments are equal i.e the magnitude and direction of their vector moments must be the same and this can only be so, if the couples act in parallel planes.

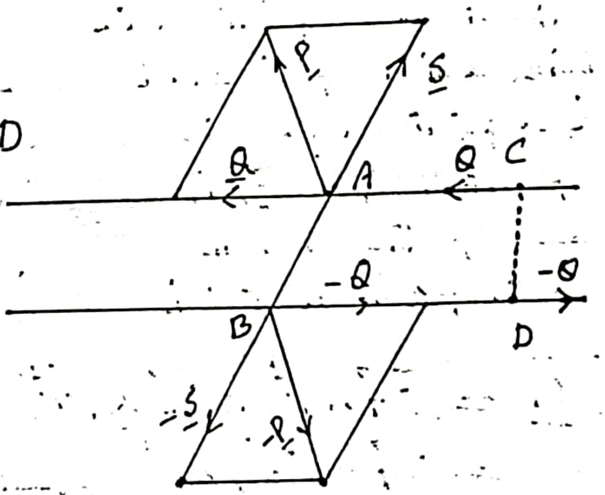
Couple arm#

A line segment which ends at the line of action of forces of couples and is perpendicular to the lines of action is called couple arm.

Equivalent Couples#

Theorem# The effect of a couple upon a rigid body is unaltered if it is replaced by any other couple of the same moment lying in the same plane.

Proof# Let $(\underline{P}, -\underline{P})$ be a couple and C, D be two arbitrary points in the plane of couple. Through C, D draw two parallel lines which intersect the lines of action of $\underline{P}$, $-\underline{P}$ at A & B respectively.



By the law of transmissibility of forces. We can regard $\underline{P}$ to act at A, and $-\underline{P}$ at B.

By the parallelogram law of forces.

$$\underline{P} = \underline{S} + \underline{Q} \quad \text{acts at A}$$
$$-\underline{P} = -\underline{S} - \underline{Q} \quad \text{acts at B}$$

The forces $\underline{S}$ $-\underline{S}$ are collinear and cancel each other and we get couple $(\underline{Q}, -\underline{Q})$ acting along

parallel lines through C, D.
Now moment of couple $(\underline{P}, -\underline{P}) = \vec{BA} \times \vec{P}$

$$= \vec{BA} \times (\vec{S} + \vec{Q})$$
$$= \vec{BA} \times \vec{S} + \vec{BA} \times \vec{Q}$$
$$= \underline{0} + \vec{BA} \times \vec{Q} \quad (\because \vec{BA} \text{ \& } \vec{S} \text{ are collinear})$$
$$= \vec{BA} \times \vec{Q}$$
$$= \text{moment of couple } (\underline{Q}, -\underline{Q})$$

By law of transmisibility $\underline{Q}, -\underline{Q}$ can be shifted at points C, D respectively. Similarly the couple $(\underline{Q}, -\underline{Q})$ can be replaced by the coplanar couple $(\underline{P}, -\underline{P})$. Hence the two couples are equivalent.

Since C, D are arbitrary points so this replacement can be done in an infinite no of ways.

Results# From this theorem we conclude that.
(1) A couple can transferred anywhere in its plane
2) It is possible to change the magnitude of the forces of a couple and its arm length arbitrarily without changing the moment of the couple.

Composition of Co-planar Couples

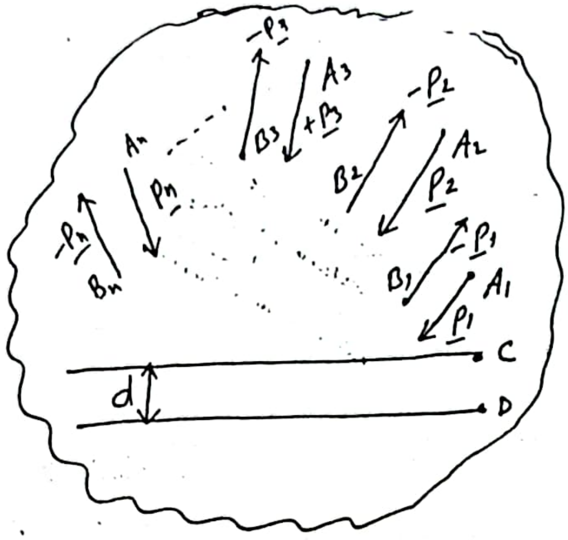
Theorem# n-Coplanar couples of moments G_i $(i = 1, 2, \ldots n)$ are equivalent to a single couple lying in the same plane whose moment G is given by

$$\underline{G} = \underline{G}_1 + \underline{G}_2 + \cdots + \underline{G}_n$$

Proof# Let the n couples $(\underline{P}_i, -\underline{P}_i)$ be such that $\underline{P}_i$ act at A_i and $-\underline{P}_i$ act at B_i. Then

$$G_i = |\vec{B_iA_i} \times \vec{P_i}|$$

We replace the couples $(\underline{P}_1 -\underline{P}_1), (\underline{P}_2, -\underline{P}_2) \dots \dots (\underline{P}_n, -\underline{P}_n)$ by equivalent couples $(\underline{Q}_1 -\underline{Q}_1), (\underline{Q}_2, -\underline{Q}_2)$ $\dots \dots (\underline{Q}_n -\underline{Q}_n)$ acting along same two parallel lines through points C, D such that forces $\underline{Q}_1, \underline{Q}_2 \dots \underline{Q}_n$ act along one line and the forces $-\underline{Q}_1 -\underline{Q}_2 \dots \dots -\underline{Q}_n$ act along a parallel lines. If d is distance between parallel lines through C, D, then we get a single couple $(\Sigma \underline{Q}_i \quad -\Sigma \underline{Q}_i)$ with length of the couple arm, (d)


$-\underline{P}_3$
A_3
B_3
$+\underline{P}_3$
$-\underline{P}_2$
A_2
B_2
$\underline{P}_2$
A_n
P_n
$-\underline{P}_n$
B_n
$-\underline{P}_1$
B_1
A_1
$\underline{P}_1$
C
d
D

The moment of this couple is

$$G = (\Sigma Q_i)d$$
$$= (Q_1 + Q_2 + \dots + Q_n)d$$
$$= Q_1 d + Q_2 d + \dots + Q_n d$$
$$= G_1 + G_2 + \dots + G_n$$

Equilibrium of a System of Co-planar Couples

The necessay and sufficent condition for the equilibrium of a system of co-planar couples is that the algebraic sum of their moments is zero

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Effect of a Couple on a Co-planar Force

Theorem# A single force and a single couple acting in the same plane upon a rigid body are equivalend to a single force acting in a direction paralled to its original direction and this single force is shifted at a distance given by $\left(\frac{\text{moment of given Couple}}{\text{Magnitude of force}}\right)$ from its original point of application

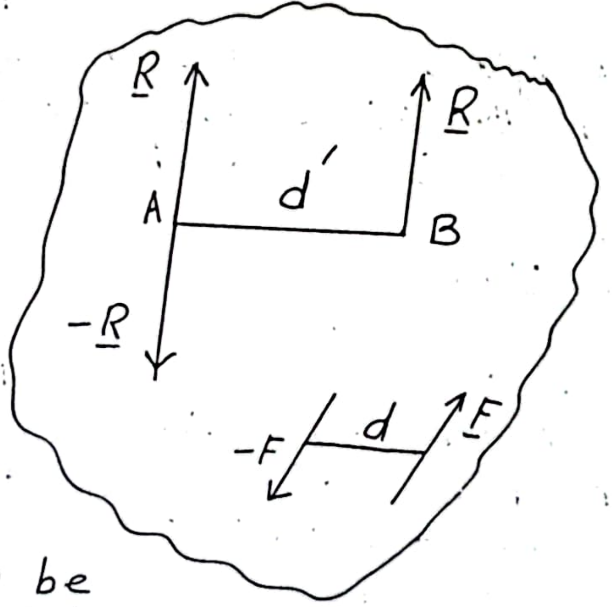
OR

The effect of a single couple G on a coplanar single force R acting on a rigid body is to shift the (point) the force R at a distance given by G/R and the direction of force remain parallel to its original direction.

Proof#

Let $\underline{R}$ be a force acting at point A of a rigid body and $(\underline{F}\ -\underline{F})$ be a Coplanar Couple acting on the rigid. If d is the length of couple arm. The moment G of couple is

$$G = Fd$$



Since a couple can be transferred anywhere in the plane, we replace the couple $(\underline{F}, -\underline{F})$ into an equivalent couple $(\underline{R}, -\underline{R})$ such that force $-\underline{R}$ acts at point A and B

Couple $(\underline{R}, -\underline{R})$.

$\because$ the couple $(\underline{F}, -\underline{F})$ & $(\underline{R}, -\underline{R})$ are equivalent.

$\therefore$

$$G = Fd = d'R$$

$$d' = G/R$$

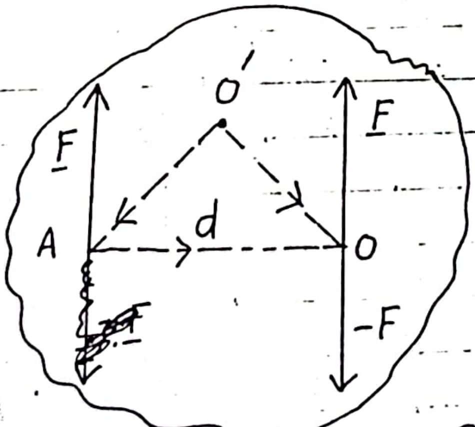
Forces $\underline{R}$; $-\underline{R}$ at A balance each other and we are left with a single force R acting at B at a distance G/R from its original position and its direction is parallel to its original direction.

From this we conclude that " a single force and a single couple acting in the same plane upon a rigid body are equivalent to a single force acting in a direction parallel to its original direction and there will be both translatory aswell as ~~rotary~~ rotatory effects.

Shifting of a Force

A force $\underline{F}$ acting on a rigid body can be be moved to any point O of the rigid body, provided a couple is added, whose moment is equal to the moment of $\underline{F}$ about O.

Proof



Suppose a force $\underline{F}$ acts at a point A of a rigid body. To shift the force $\underline{F}$ to point O of rigid body, at point O we introduce two forces $\underline{F}$, $-\underline{F}$ at O without changing the ~~original~~ effect of original force $\underline{F}$ on the body. The force $\underline{F}$ at A and $-\underline{F}$ at O form a couple $(\underline{F}, -\underline{F})$. Hence the given force $\underline{F}$ at is equivalent to a force

$\underline{F}$ at o together with a couple $(\underline{F}, \underline{F})$ whose moment is

$$Fd = \text{moment } \underline{F} \text{ at } A \text{ about } O$$

Now we prove that the force $\underline{F}$ at A is equivalent to a force $\underline{F}$ at O together with a Couple $(\underline{F}, -\underline{F})$ as.

The resultant of forces of the couple and force $\underline{F}$ at $O = \underline{F} + \underline{F} - \underline{F} = \underline{F}$

The resultant turning effect of $(\underline{F}, -\underline{F})$ and force $\underline{F}$ at O, about any point in their plane is

Resultant moment = Moment of Couple about O' + moment of $\underline{F}$ at O, about O'

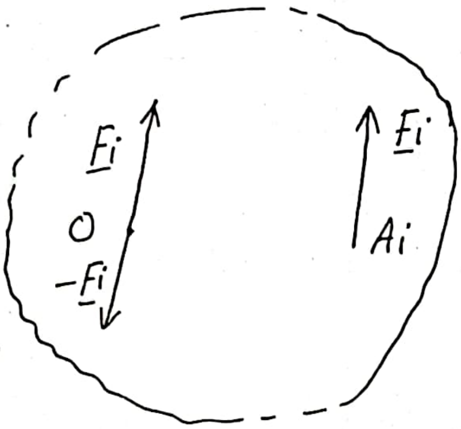
$$\begin{aligned}
&= \overrightarrow{OA} \times \vec{F} + \overrightarrow{O'O} \times \vec{F} \\
&= \overrightarrow{OA} \times \vec{F} + (\overrightarrow{O'A} + \overrightarrow{AO}) \times \vec{F} \\
&= \overrightarrow{OA} \times \vec{F} + \overrightarrow{O'A} \times \vec{F} + \overrightarrow{AO} \times \vec{F} \\
&= \overrightarrow{OA} \times \vec{F} + \overrightarrow{O'A} \times \vec{F} - \overrightarrow{OA} \times \vec{F} \\
&= \overrightarrow{O'A} \times \vec{F} \\
&= \text{Moment of } \underline{F} \text{ at } A \text{ about } O'
\end{aligned}$$

Thus the turning effect of the Couple and $\underline{F}$ at O about any point in their plane is same as the turning effect of $\underline{F}$ at A about that point. Also the the forces of Couple and force $\underline{F}$ at O is equal to $\underline{F}$. Therefore the Couple $(\underline{F}, -\underline{F})$ and force $\underline{F}$ at O are together equivalent to force $\underline{F}$ at A i.e effect of $\underline{F}$ is not changed in any respect and it is shifted at point O

Reduction of a System of Coplanar Forces to one Force and One Couple

Theorem# A system of coplanar forces acting upon a rigid body can be reduced to a single force acting at an arbitary point in the plane of forces together

Proof# Suppose forces $\underline{F}_i$ $(i = 1, 2, \cdots n)$ in plane act upon a rigid body at points A_i. We transfer the forces at point O by introducing couples of moments $\underline{G}_1, \underline{G}_2 \cdots \underline{G}_n$. As such we get a system of forces $\underline{E}_1, \underline{E}_2 \cdots \underline{E}_n$ acting at O and a system of coplanar couples of moments $\underline{G}_1, \underline{G}_2, \cdots \underline{G}_n$.



The resultant $\underline{R}$ of the forces is given by

$$\underline{R} = \underline{F}_1 + \underline{E}_2 + \underline{E}_3 + \cdots + \underline{E}_n$$

$\underline{R}$ passes through O.

By the Composition of couples, all the coplanar couples can be replaced by a single couple of moment $\underline{G}$

$$\underline{G} = \underline{G}_1 + \underline{G}_2 + \cdots + \underline{G}_n$$

$$\underline{G}_i = \overrightarrow{OA_i} \times \vec{F}_i$$

Depending upon the value of $\underline{R}$ & $\underline{G}$ following are some particular cases:

(1) **Case of Equilibrium#**

If $\underline{R} = \underline{0}$, $\underline{G} = \underline{0}$

which shows that does not affect translatory and rotatory motion of the body and hence given system is in equilibrium.

Case # 2 Non-Equilibrium Cases

(i) If $\underline{R} = \underline{0}$, $\underline{G} \neq \underline{0}$, then system is equivalent to a couple

(ii) If $\underline{R} \neq \underline{0}$, then system can be reduced to a single forces. Two cases are possible

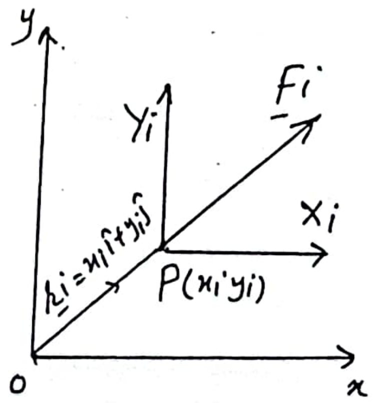
(a) $\underline{R} \neq \underline{0}$, $\underline{G} = \underline{0}$, then the system of forces is equivalent to a single force and $\underline{G} = \underline{0}$ shows that the system has no rotatory effect.

(b) If $\underline{R} \neq \underline{0}$, $\underline{G} \neq \underline{0}$, then the couple shifts the resultant force $\underline{R}$ through a distance G/R and direction of force remains parallel to its original direction. But here both translatory and rotatory effect will be present.

These cases of non-equilibrium show that If a coplanar system of forces is not in equilibrium, it can be reduced either a single force (when $\underline{R} \neq \underline{0}$) or a single couple (when $\underline{R} = \underline{0}$)

Reduction of Coplanar Forces

(Analytic Method)



Let the forces $\underline{F}_i$ $(i = 1, 2, \cdots n)$ act at points $P_i(x_i, y_i)$. Let X_i, Y_i are the resolved parts of the forces along x-axis and y-axis respectively. Then

$$\underline{F}_i = X_i\hat{i} + Y_i\hat{j} \longrightarrow ①$$

$$r_i - x_i\hat{i} + y_i\hat{j} \longrightarrow ②$$

If we reduce (shift) the forces at O(0,0), then n couples of moments $\underline{G}_1, \underline{G}_2 \cdots \underline{G}_n$ are also introduced and we get n concurrent forces whose resultan $\underline{R}$ is

$$\underline{R} = \sum_{i=1}^{n} \underline{F}_i = \sum_{i=1}^{n} (X_i\hat{i} + Y_i\hat{j})$$

And n couples $\underline{G}_i$ $(i=1,2,\cdots n)$ are equivalent to single couple whose moment $\underline{G}$ is sum of moments of all the forces about O i-e

$$\underline{G} = \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i$$

$$= \sum_{i=1}^{n} \underline{G}_i$$

$$= \sum_{i=1}^{n} (x_i\hat{i} + y_i\hat{j}) \times (X_i\hat{i} + Y_i\hat{j})$$

$$= \sum_{i=1}^{n} (x_iY_i - y_iX_i)\hat{k}$$

where $\hat{k}$ is unit vector perpendicular to forces.

If X and Y are resolved parts resultant at O, Then

$$\underline{R} = X\hat{i} + Y\hat{j} = \left(\sum_{i=1}^{n} X_i\right)\hat{i} + \left(\sum_{i=1}^{n} Y_i\right)\hat{j}$$

Comparing co-efficients of $\hat{i}, \hat{j}$

$$X = \sum_{i=1}^{n} X_i \qquad Y = \sum_{i}^{n} Y_i$$

$$R = \sqrt{X^2 + Y^2}$$

If θ is the angle of resultant with x-axis, Then

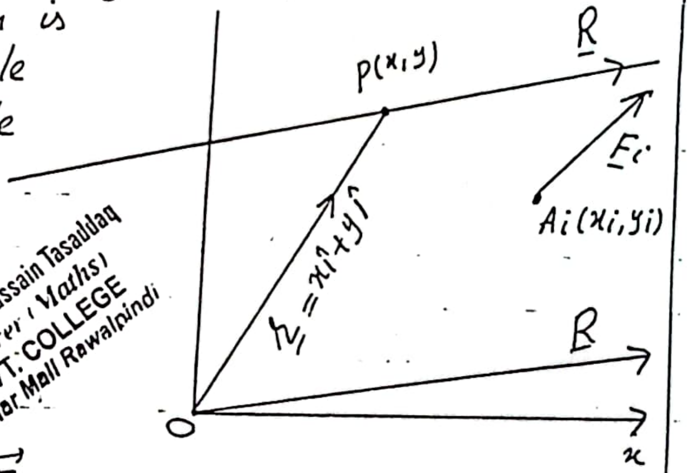
$$\tan\theta = Y/X \Rightarrow \theta = \tan^{-1}(Y/X)$$

Note Once the system of coplanar forces has been reduced to a force $\underline{R}$ and at Couple $\underline{G}$ at any point O, it can easily be reduced to a force $\underline{R}$ at another point O′ and a couple. While the resultant $\underline{R}$ will remain unchanged, the new couple will have moment $\underline{G}'$ equal to the algebric sum of the moments of the forces about O′ or its moment $\underline{G}'$ is equal to the sum of moment $\underline{G}$ and moment of couple introduced due to shifting of the resultant force $\underline{R}$ at other point.

Equation of Line of action of Resultant

Consider a system of n-co-planar forces F_i $(i=1,2,\dots n)$ acting at $A_i (i=1,2\dots n)$.

Suppose the system is reduced into a single force $\underline{R}$ and single couple of moment $\underline{G}$ at origin. Then



$$\underline{R} = \sum_{i=1}^{n} F_i$$

and

$$\underline{G} = \sum_{i=1}^{n} \overrightarrow{OA_i} \times \overrightarrow{F_i}$$

Let the resultant $\underline{R}$ be shifted to a new point in the plane and P(x,y) be any point on the line of action of the resultant in new position. The magnitude & direction of $\underline{R}$ remains same but but the moment of the resultant couple $\underline{G}'$ is is equal to the sum of the moment of couple $\underline{G}$ and moment of the couple which is added due to shifting of R i.e

$\underline{G}'$ = Sum of moments of all the forces about $P(x,y)$

= $\underline{G}$ + Moment of new Couple introduced due to shifting of $\underline{R}$

= $\underline{G}$ + Moment of $\underline{R}$ about any on the line of action of $\underline{R}$ in new position

$$= \underline{G} + (-\underline{r} \times \underline{R})$$

$$\underline{G}' = \underline{G} + \underline{R} \times \underline{r}$$

As the forces lie in xy-plane so $\underline{G}, \underline{G}'$ have same directions along the +ve z-axis considering the rotational effect counterclockwise.

Hence

$$G'\hat{k} = \underline{G} + (X\hat{i}+Y\hat{j}) \times (x\hat{i}+y\hat{j})$$

$$= G\hat{k} + Xy\hat{k} - xY\hat{k}$$

$$= (G - xY + yX)\hat{k}$$

Since the point $P(x,y)$ lies on the line of action of the resultant, the sum of moments of all the forces about $P(x,y)$ must be zero.

$$\underline{G}' = \underline{0}$$

$$\Rightarrow (G - xY + yX)\hat{k} = 0$$

$$\Rightarrow G - xY + yX = 0 \longrightarrow Ⓐ$$

Results # (i) If the resultant $\underline{R}$ passes through $(0,0)$, then from Ⓐ

$$G - 0Y + 0X = 0 \quad \Rightarrow G = 0$$

Equation of the line of the resultant becomes

$$-xY + yX = 0$$

(b) Since $G' = G - xY + yX$ is the sum of moments of all the forces about $P(x, y)$, therefore to find the sum of moments of forces about any point we put the co-ordinates of that point in this identity after the reduction has been made at origin

(c) When the system of coplanar forces is in equilibrium, we have

$$X = 0 \quad Y = 0 \quad , G = 0$$

Hence

$$G' = G - 0 + 0 = G$$

i.e the sum of the moments of all the forces about any point must be zero

(d) $\underline{R} = \underline{0} \quad \underline{G} \neq \underline{0}$ i.e $X = 0 = Y, \; G \neq 0$

Then from

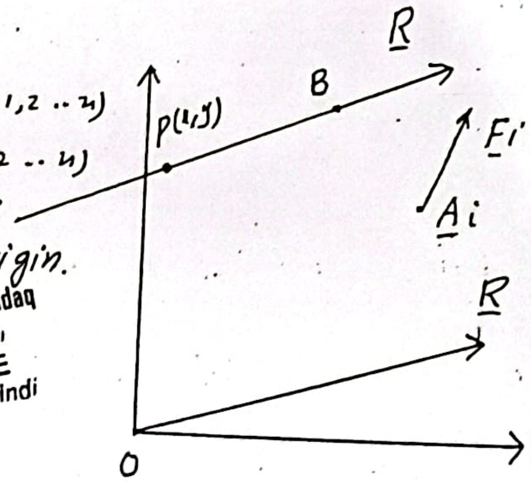
$$G' = G - xY + yX$$

$$G' = G$$

i.e the sum of moments of all the forces about different points must have non-zero equal values.

Second Method

Suppose the system of n- co-planar forces $\underline{F}_i \; (i = 1, 2 .. n)$ acting at points $A_i \; (i = 1, 2 .. n)$ is reduced into one force $\underline{R}$ and a couple $\underline{G}$ at origin. O. Then



$$\underline{R} = \Sigma \underline{F}_i$$

$$\underline{G} = \sum_{i=1}^{n} \overrightarrow{OA_i} \times \underline{F}_i$$

When the reduction is made into a resultant $\underline{R}$ and a couple at point B, then the sum of moments of all the forces about any point $P(x, y)$ on the line action of the resultant is

$$\underline{G}'' = \text{Moment of couple when forces are reduced at } B + \overrightarrow{PB} \times \Sigma \underline{F}_i$$

$$= \Sigma \overrightarrow{BA_i} \times \vec{F_i} + \overrightarrow{PB} \times \Sigma \vec{F_i}$$

$$= \Sigma(\overrightarrow{OA_i} - \overrightarrow{OP} - \overrightarrow{PB}) \times \vec{F_i} + \overrightarrow{PB} \times \Sigma \vec{F_i}$$

$$\overrightarrow{OA_i} = \overrightarrow{OP} + \overrightarrow{PB} + \overrightarrow{BA_i}$$

$$\overrightarrow{BA_i} = \overrightarrow{OA_i} - \overrightarrow{OP} - \overrightarrow{PB}$$

$$= \Sigma \overrightarrow{OA_i} \times \vec{F_i} - \overrightarrow{OP} \times \Sigma \vec{F_i} + \overrightarrow{PB} \times \Sigma \vec{F_i} + \overrightarrow{PB} \times \Sigma f_i$$

$$= \Sigma \overrightarrow{OA_i} \times \vec{F_i} - \overrightarrow{OP} \times \Sigma \vec{F_i}$$

$$= \underline{G} + \overrightarrow{PO} \times \vec{R}$$

$$= G\hat{k} + (-xY + yX)\hat{k}$$

$$= (G - xY + yX)\hat{k}$$

$\because$ point (x, y) lies on the line of action of the resultant.

$\therefore$ $\underline{G}'' = \underline{0}$

$\Rightarrow$ $(G - xY + yX)\hat{k} = \underline{0}$

$\Rightarrow$ $G - xY + yX = 0$

Third Method

Let a system of coplanar forces be reduced into a single resultant force $\underline{R}$ and a single couple $\underline{G}$ at origin. Coplanar couple $\underline{G}$ shifts the force $\underline{R}$ along OB parallel to its original direction such that

$$\overline{OO'} = \frac{G}{R}$$

$\because$ Since $\tan\theta = Y/X$

Therefore $\sin\theta = Y/R$. $\cos\theta = X/R$

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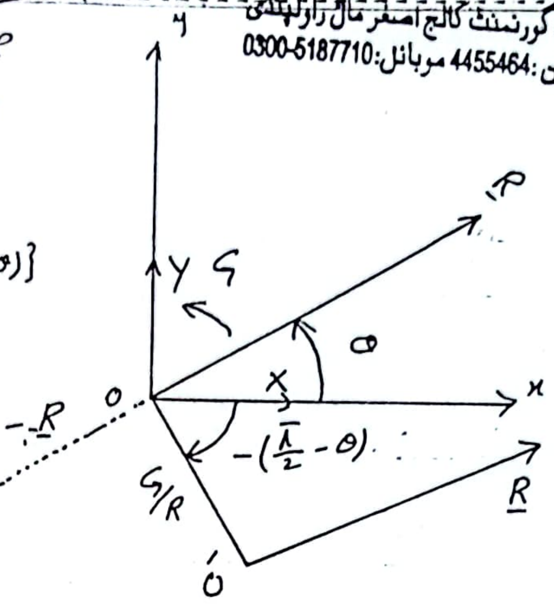
The equation of line of action of the resultant in normal form is



$$x\cos\{-(\pi/2-\theta)\}+y\sin\{-(\pi/2-\theta)\} = \overline{OO'}$$

$$x\sin\theta - y\cos\theta = G/R$$

$$x\frac{Y}{R} - y\frac{X}{R} = G/R$$

$$xY - yX = G$$

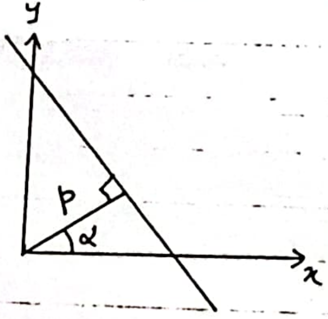
$$\Rightarrow G - xY + yX = 0$$

Note # (1) $\underline{R}$ is shifted downward so that turning effect of $\underline{R}$ is anticlockwise which also the direction of turning effect couple $\underline{G}$ which shifts $\underline{R}$ through distance G/R. Also note that net linear and turning effect of $\underline{R}$ at O' is equivalent to together $\underline{R}$ at O and couple $\underline{G}$

(2) Equation of a line in normal form is

$$x\cos\alpha + y\sin\alpha = p$$

where p is length of perpendicular from origin to line and α is its angle with x-axis.



The Location of Lines in Space

The location of two lines in space may be such that

(a) the lines are parallel

(b) the lines are not parallel and intersect

(c) the lines are parallel and do not intersect

Skew Lines

Two lines which are neither

parallel nor they intersect are called skew lines.

Any two skew lines are non-coplanar and lie in parallel planes.

Reduction of a System of non-Coplanar Forces

We know that a system of coplanar forces is (either) equivalent ~~to~~ either to a ~~str~~ single force or to a couple or that the system is in equilibrium. But if we consider a set of non-coplanar forces a fourth possibility exists. Consider, for example, two forces whose lines of action are skew. Such a pair of forces cannot be in equilibrium and they cannot reduce either to a single force or to a single couple but are equivalent to a combination of a force and couple, where the plane of couple does not contain the force.

Thus a a set of non-coplanar forces is equivalent to

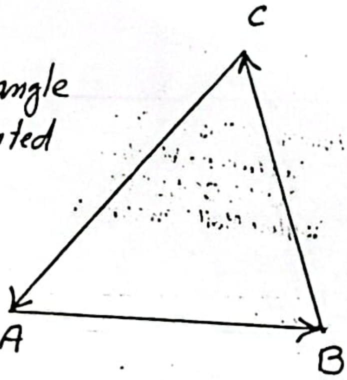
either (a) a single force.
or (b) a couple
or (c) a combination of a couple and non-coplanar force (when the lines of action are skew)
or (d) the system is in equilibrium.

Note # To prove that a system of forces reduces to a single force is not so straight forward in general. If for a given system $\Sigma \underline{F} \neq \underline{0}$, the system could reduce to either a single force or a combination of a couple and a non-coplanar force. However there are some cases when we can rule out the possibility of a resultant force and a non-coplanar couple. If the lines of action of the forces are Concurrent, all passing through the point A say, then the vector sum of the moments of the forces about A is zero and the system must reduce to a single force. Alternatively, if the forces are coplanar, the resultant system will act in the same plane thus eliminating the possibility of non-coplanar couple.

Example # If three forces are represented in magnitude, direction and position by sides of triangle taken in order, they are equivalent.

Sol # Let $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ be Co-planar forces acting along the corresponding sides of a triangle ABC and are completely represented by sides.

When the forces are reduced to a point A, their resultant is



$$\underline{R} = \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\overrightarrow{0}$$ (By triangle law of forces)

$\underline{G}$ = moment of the resultant couple
= Sum of moments of all the forces about A
$= \overrightarrow{AB}\times\overrightarrow{BC}+\overrightarrow{0}$

⇒ Coplanar forces are equivalent to a couple with:

$$|\underline{G}| = |\overrightarrow{AB}\times\overrightarrow{BC}| = 2\cdot\left|\tfrac{1}{2}\,\overrightarrow{AB}\times\overrightarrow{BC}\right|$$
$= 2$ (area of the triangle ABC)

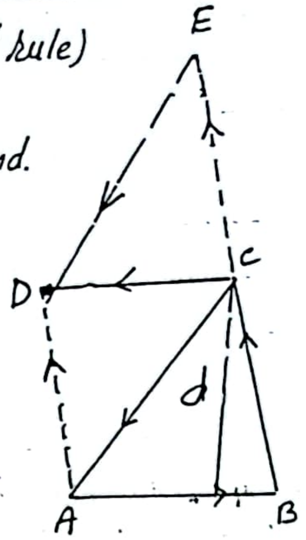
OR

By parallelogram law of forces

$$\overrightarrow{CD} = \overrightarrow{CA}+\overrightarrow{BC}$$

$= \overrightarrow{BA}$ (By head-to-tail rule)

$\overrightarrow{CD} = -\overrightarrow{AB}$



Hence we have two forces $\overrightarrow{CD}$ and $\overrightarrow{AB}$ acting at C & A in opposite sense. Therefore the system reduces to a couple $(\overrightarrow{CD}, -\overrightarrow{AB})$. If (d) is the length of ⊥ar from C to AB, then

Magnitude of the moment of

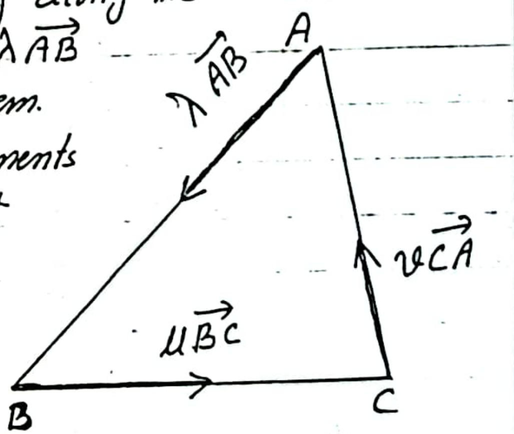
$$\text{Couple} = d\,AB$$
$$= 2\left(\tfrac{1}{2}\,d\,AB\right)$$
$$= 2(\text{Area of } \Delta ABC)$$

OR

$$\text{Moment of Couple} = |\overrightarrow{CA} \times \overrightarrow{AB}|$$
$$= 2\left|\tfrac{1}{2}\overrightarrow{CA} \times \overrightarrow{AB}\right|$$
$$= 2(\text{area of } \Delta ABC)$$

Example # Forces act along sides BC, CA, AB of triangle. Show that they are equivalent to a couple only if the forces are proportional to the sides.

Sol # Let the forces acting along the sides of the triangle be $\lambda\overrightarrow{AB}$, $\mu\overrightarrow{BC}$, $\nu\overrightarrow{CA}$. Since the system reduces to a couple, the moments of the forces about any point remain the same.



Sum of moments about A

= Sum of moments about B

= Sum of moments about C

$$|\overrightarrow{AB} \times \mu \overrightarrow{BC}| = |\overrightarrow{BC} \times \nu \overrightarrow{CA}| = |\overrightarrow{CA} \times \lambda \overrightarrow{AB}|$$

$$\Rightarrow \mu\, |\overrightarrow{AB} \times \overrightarrow{BC}| = \nu\, |\overrightarrow{BC} \times \overrightarrow{CA}| = \lambda\, |\overrightarrow{CA} \times \overrightarrow{AB}|$$

$$\Rightarrow \mu(2\Delta) = \nu(2\Delta) = \lambda(2\Delta)$$

$$\Rightarrow \mu = \nu = \lambda$$

$\Rightarrow$ Forces are $\lambda \overrightarrow{AB}, \lambda \overrightarrow{BC}, \lambda \overrightarrow{CA}$

$\Rightarrow$ i.e the forces are proportional to the sides of the triangle

---

Example # If forces $l\overrightarrow{AB}, m\overrightarrow{BC}, l\overrightarrow{CD}, m\overrightarrow{DA}$ acting along the sides of a quadrilateral are equivalent to a couple, show that either $l = m$ or ABCD is a parallelogram.

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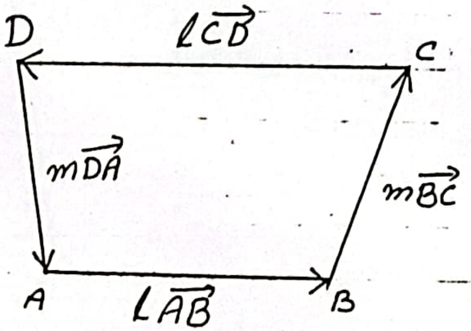
Sol #

Clearly, the forces are co-planar.

Since the system reduces to a couple, the sum of moments of all the forces about any point remain same



Sum of moments of all the forces about A = Sum of moments of all the forces about C.

$$\Rightarrow \overrightarrow{AB} \times m\overrightarrow{BC} + \overrightarrow{AD} \times l\overrightarrow{CD} = \overrightarrow{CD} \times m\overrightarrow{DA} + \overrightarrow{CB} \times l\overrightarrow{AB}$$

$$\Rightarrow m\, \overrightarrow{AB} \times \overrightarrow{BC} - m\, \overrightarrow{CD} \times \overrightarrow{DA} - \overrightarrow{DA} \times l\overrightarrow{CD} - \overrightarrow{CB} \times l\overrightarrow{AB} = \vec{0}$$

$$\Rightarrow m\, \overrightarrow{AB} \times \overrightarrow{BC} - m\, \overrightarrow{CD} \times \overrightarrow{DA} - l\, \overrightarrow{DA} \times \overrightarrow{CD} + \overrightarrow{BC} \times l\overrightarrow{AB} = \vec{0}$$

$$\Rightarrow m\, \overrightarrow{AB} \times \overrightarrow{BC} - m\overrightarrow{CD} \times \overrightarrow{DA} - l\, \overrightarrow{DA} \times \overrightarrow{CD} - l\, \overrightarrow{AB} \times \overrightarrow{BC} = \vec{0}$$

$$\Rightarrow m\, \overrightarrow{AB} \times \overrightarrow{BC} - m\, \overrightarrow{DA} \times \overrightarrow{DC} + l\, \overrightarrow{DA} + \overrightarrow{DC} + l\, \overrightarrow{BA} \times \overrightarrow{BC} = \vec{0}$$

$$\Rightarrow -m\,\overrightarrow{BA}\times\overrightarrow{BC} - m\,\overrightarrow{DA}\times\overrightarrow{DC} + l\,\overrightarrow{DA}\times\overrightarrow{DC} + l\,\overrightarrow{BA}\times\overrightarrow{BC} = \vec{0}$$

$$(l-m)\,\overrightarrow{BA}\times\overrightarrow{BC} + (l-m)\,\overrightarrow{DA}\times\overrightarrow{DC} = \vec{0}$$

$$(l-m)\left[\overrightarrow{BA}\times\overrightarrow{BC} + \overrightarrow{DA}\times\overrightarrow{DC}\right] = \vec{0}$$

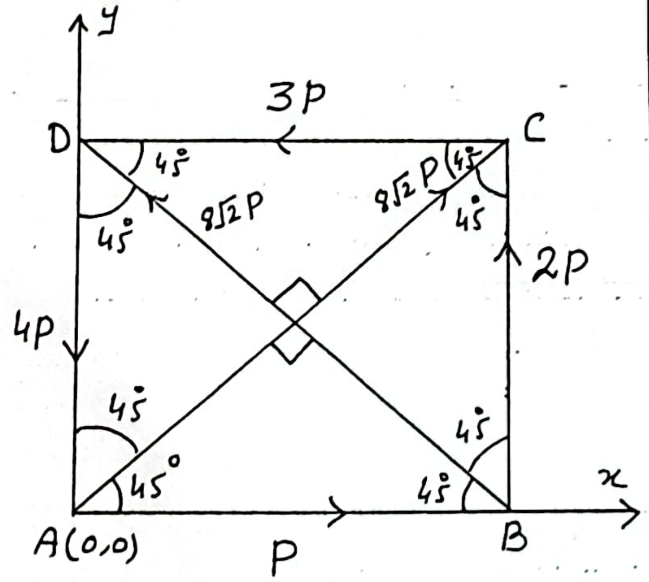
$$\Rightarrow l-m=0 \quad \text{or} \quad \overrightarrow{BA}\times\overrightarrow{BC} + \overrightarrow{DA}\times\overrightarrow{DC} = \vec{0}$$

$$\Rightarrow l=m \quad \text{or} \quad \overrightarrow{BA}\times\overrightarrow{BC} = \overrightarrow{DC}\times\overrightarrow{DA}$$

$\Rightarrow l = m$ or ABCD is a parallelogram.

Example # Forces of magnitude $P, 2P, 3P, 4P$ act respectively along the sides AB, BC, CD, DA of a square ABCD, of side a, and forces each of magnitude $(8\sqrt{2})P$ act along the diagonals BD, AC. Find the magnitude of the resultant force and the distance of its line of action from A.

Sol # Let the side AB of square be along x-axis and AD along y-axis. Let X, Y be the resolved parts of the resultant along x-axis and y-axis respectively.



Resolving the forces along x-axis and y-axis

$$X = P\cos 0^\circ + 2P\cos 90^\circ + 3P\cos 180^\circ + 4P\cos(-90^\circ) + 8\sqrt{2}P\cos(90^\circ+45^\circ) + 8\sqrt{2}P\cos 45^\circ$$

$$= P + 0 - 3P + 0 - 8\sqrt{2}P\sin 45^\circ + 8\sqrt{2}P\cdot\frac{1}{\sqrt{2}}$$

$$= P - 3P - 8\sqrt{2}P\cdot\frac{1}{\sqrt{2}} + 8P = -2P$$

$$Y = P\sin 0^\circ + 2P\sin 90^\circ + 3P\sin 180^\circ + 4P\sin(-90^\circ) + 8\sqrt{2}P\sin(90^\circ+45^\circ) + 8\sqrt{2}\sin 45^\circ$$

$$= P(0) + 2P + 3P(0) - 4P + 8\sqrt{2}P\cos 45^\circ + 8\sqrt{2}P\cdot\frac{1}{\sqrt{2}}$$

$$= 0 + 2P + 0 - 4P + 8\sqrt{2}P\cdot\frac{1}{\sqrt{2}} + 8P$$

$$= 14P$$

The magnitude of the resultant is

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$$R = \sqrt{X^2+Y^2}$$

$$= \sqrt{(-2P)^2 + (14P)^2} = \sqrt{4P^2+196P^2}$$

$$= \sqrt{200}\,P = 10\sqrt{2}\,P$$

Taking moment about A, the moment of the resultant Couple is

G = The sum of moments of all the forces, about A.

$$= 2P(a) + 3P(a) + 8\sqrt{2}\,P\,a\sin 45^\circ$$

$$= 2ap + 3ap + 8\sqrt{2}\,Pa\cdot\frac{1}{\sqrt{2}}$$

$$= 13ap$$

Equation of the line of action of the resultant is

$$G - xY + yX = 0$$

putting $G = 13ap$ $\quad X = -2P \quad Y = 14P$

$$13ap - x(14P) + y(-2P) = 0$$

$$13a - 14x - 2y = 0$$

$$13a = 2(7x+y)$$

Distance of the line of action of the resaltant from $A(0,0)$ $= \dfrac{|13a|}{\sqrt{(14)^2+(2)^2}} = \dfrac{13a}{\sqrt{200}} = \dfrac{13a}{10\sqrt{2}}$

Geometry About Square

(1) The diagonals of a square are Congruent and perpendicular bisectors of each other

(2) The diagonals of a square are bisectors of its angles

Note # If two equal line segments bisect each other, then the four segments made are all equal in length

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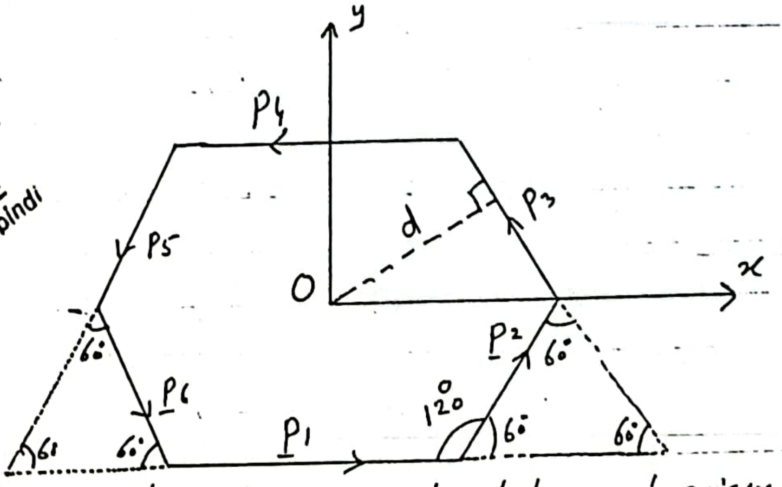
Example # Forces $P_1, P_2, P_3, P_4, P_5, P_6$ act along the sides of a regular hexagon taken in order. Show that they will be in equilibrium if

$$\Sigma P = 0 . \quad P_1 - P_4 = P_3 - P_6 = P_5 - P_2$$

Sol #



Let the centre of the hexagon be taken at origin and X, Y are resolved parts of the resultant

$$X = P_1 \cos 0^\circ + P_2 \cos 60^\circ + P_3 \cos 120^\circ + P_4 \cos 180^\circ + P_5 \cos(180^\circ + 60^\circ)$$
$$+ P_6 \cos \cos(-60^\circ)$$
$$= P_1 + P_2(1/2) + P_3 \cos(90^\circ + 30^\circ) - P_4 - P_5 \cos 60^\circ + \tfrac{1}{2} P_6$$
$$= P_1 + \tfrac{1}{2} P_2 - \tfrac{1}{2} P_3 - P_4 - \tfrac{1}{2} P_5 + \tfrac{1}{2} P_6$$
$$= P_1 - P_4 + \tfrac{1}{2}(P_2 - P_5) + \tfrac{1}{2}(P_6 - P_3) \longrightarrow ①$$

$$Y = P_1 \sin 0^\circ + P_2 \sin 60^\circ + P_3 \sin 120^\circ + P_4 \sin 180^\circ$$
$$+ P_5 \sin(180^\circ + 60^\circ) + P_6 \sin(-60^\circ)$$

$$Y = 0 + \frac{\sqrt{3}}{2}P_2 + \frac{\sqrt{3}}{2}P_3 + 0 + \frac{\sqrt{3}}{2}P_5 - \frac{\sqrt{3}}{2}P_6$$

$$= \frac{\sqrt{3}}{2}(P_2 - P_5) + \frac{\sqrt{3}}{2}(P_3 - P_6) \longrightarrow ②$$

If d is perpendicular distance of each side from origin (Centre)

G = The sum of moments of forces about o

$$= (P_1 + P_2 + P_4 + P_3 + P_5 + P_6)\,d$$

$$= \sum_{i=1}^{6} P_i d \longrightarrow ③$$

The forces are in equilibrium if $G = 0$ $\quad X = 0 \quad Y = 0$

$$X = 0 \Rightarrow P_1 - P_4 + \tfrac{1}{2}(P_2 - P_5) + \tfrac{1}{2}(P_6 - P_3) = 0 \rightarrow ④$$

$$Y = 0 \Rightarrow$$

$$\frac{\sqrt{3}}{2}(P_2 - P_5) + \frac{\sqrt{3}}{2}(P_3 - P_6) = 0$$

$$P_2 - P_5 = -(P_3 - P_6) = P_6 - P_3$$

Putting in ④

$$P_1 - P_4 + \tfrac{1}{2}(P_2 - P_5) + \tfrac{1}{2}(P_2 - P_5) = 0$$

$$P_1 - P_4 + P_2 - P_5 = 0$$

$$P_1 - P_4 = P_5 - P_2$$

Also $\quad P_5 - P_2 = P_3 - P_6$ from ⑤

$$\Rightarrow P_1 - P_4 = P_5 - P_2 = P_3 - P_6$$

Also

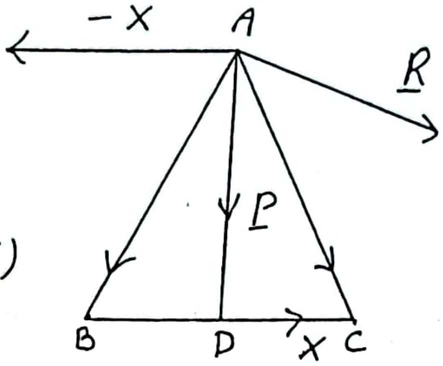
$$\Sigma P d = 0$$

$$\Rightarrow \Sigma P = 0 \quad \because d \neq 0$$

Example # Prove that any system of forces in a plane is equivalent to three suitably chosen forces X, Y, Z acting along the sides BC, CA, AB of a given triangle in the plane. Prove that if the system is equivalent to a Couple G, then

$$\frac{X}{BC} = \frac{Y}{CA} = \frac{Z}{AB} = \frac{G}{2\Delta}$$

Sol #



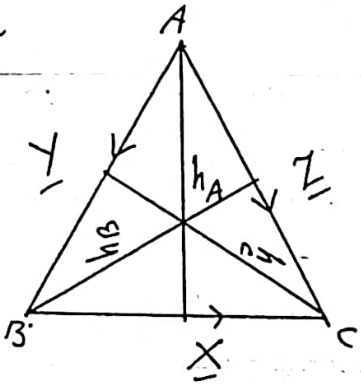
Let the system is reduced at point A into a resultant force $\underline{R}$ and a resultant Couple $\underline{G}$. Replace the forces of Couple in $(\underline{X}, -\underline{X})$ such that force X is acting along BC and $-\underline{X}$ is acting at A. Since forces $-\underline{X}$, $\underline{R}$ are concurrent at A, their resultant can be found. Let $\underline{P}$ be their resultant. Let the line of action $\underline{P}$ meets BC at pt D. Then

$$\underline{P} = (\lambda+\mu)\overrightarrow{AD} \qquad (\because \underline{P} \text{ and } \overrightarrow{AD} \text{ are Collinear})$$

$$= \lambda\overrightarrow{AB} + \mu\overrightarrow{AC} \qquad (\text{By } \lambda\text{-}\mu \text{ Theorem})$$

where $\overline{BD} : \overline{DC} = \mu : \lambda$

Hence force system is reduced into three forces $\underline{X}$, $\underline{Y} = \lambda\overrightarrow{AB}$ $\underline{Z} = \mu\overrightarrow{CA}$ acting along sides BC, AB & AC of ΔABC.



Let h_A, h_B, h_C be the perpendicular distances of forces $\underline{X}$ $\underline{Z}$, $\underline{Y}$ from A B & C respectively.

Since system is equivalent to a Couple, Sum of moments of forces about any point will be Constant say G.

Sum of moment about A = Sum of moments about B = Sum of moments about C

$$X h_A = Z h_B = Y h_C = G \qquad \rightarrow ①$$

Now

$h_A \cdot BC = 2\Delta$ $\qquad h_B \cdot AC = 2\Delta$ $\qquad h_C \cdot AB = 2\Delta$

$$h_A = \frac{2\Delta}{BC} \qquad h_B = \frac{2\Delta}{AC} \qquad h_C = \frac{2\Delta}{AB}$$

putting these values in ①

$$\frac{X \cdot 2\Delta}{BC} = \frac{Z \cdot 2\Delta}{AC} = \frac{Y \cdot 2\Delta}{AB} = G$$

$$\Rightarrow \frac{X}{BC} = \frac{Z}{AC} = \frac{Y}{AB} = \frac{G}{2\Delta}$$

Hexagon # A polygon of six sides is called hexagon.

Hexagon is regular if its all sides are equal and its interior angles are all equal, each equal to 120°

Condition of Equilibrium of System of Co-planar Forces

Since a system of coplanar forces reduces to a single force $\underline{R}$ acting at an arbitrary point of the plane of the forces together with a couple G, therefore a system of coplanar forces is said to be in equilibrium if

$$\underline{R} = \underline{0} \quad \& \quad \underline{G} = \underline{0}$$

$$\Rightarrow X = 0 \qquad Y = 0 \qquad G = \Sigma(x_i Y_i - y_i X_i) = 0$$

Thus a system of coplanar forces is in equilibrium if the algebraic sum of the resolved parts of the forces in any two directions (two directions need not be at right angles but they must not be parallel) is zero and the algebraic sum of the moments of all the forces about any arbitrary point is zero

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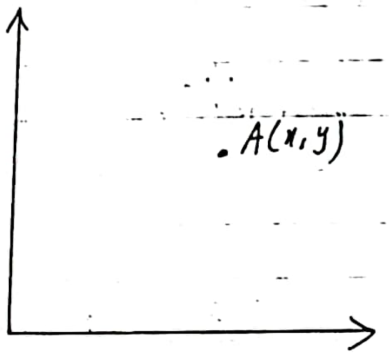
Theorem # A system of coplanar forces is in equilibrium if the algebraic sum of moments of all the forces about two different points O and A is zero and the algebraic sum of the resolved parts in some one direction, not perpendicular to OA is zero

Proof # Let O be taken as origin and Co-ordinates of A be (x, y). The direction of x-axis is not perpendicular to OA.

Since OA is not $\perp$ar to x-axis, $x \neq 0$.

Let X, Y be the algebraic sums of the resolved parts of the forces and G, G′ be the algebraic sum of moment of forces about O, A:

Then



$$G' = G - xY + yX \rightarrow ①$$

According to given Condition,

$$G = 0 \text{ and } X = 0, G' = 0$$

using these in ①

$$\therefore 0 = 0 - xY + 0$$

$$\Rightarrow xY = 0$$

$$\Rightarrow Y = 0 \qquad \because x \neq 0$$

Thus $X = Y = 0$ and $G = 0$

$\Rightarrow$ the forces are in equilibrium.

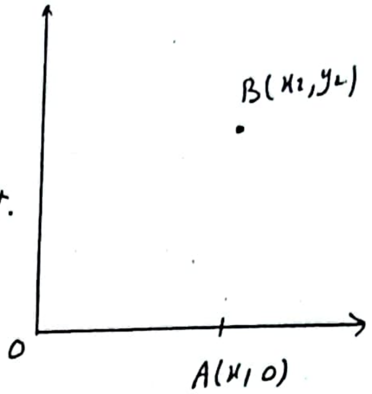
Theorem # A system of coplanar forces is in equilibrium if the algebraic sum of the moments of all the forces about three different points O, A, B, not in the same straight line (non-collinear), are separately zero.

Proof # Let O be origin and A, B be $(x_1, 0)$, (x_2, y_2).

$\because$ O, A, B are not collinear

$\therefore \quad y_2 \neq 0$

Let G, G_1, G_2 be the algebraic sum of all the moments of all the forces about O, A, B respectively. Let X, Y be the resolved parts of the resultant.



Sum of moments of all the forces about $O = G$

Sum of the moments of all the forces about $A = G_1 = G - x_1 Y \rightarrow ①$

Sum of the moments of all the forces about B

$= G_2 = G - x_2 Y + y_2 X \longrightarrow ②$

According to given condition.

$G = 0 \qquad G_1 = 0 \quad , \quad G_2 = 0$

using these in ① & ②

from ① we get

$$G - x_1 Y = 0$$

$$x_1 Y = 0$$

$$Y = 0 \qquad \because x_1 \neq 0$$ because O, A, B are different points

from ②

$$0 = 0 - x_2(0) + y_2 X$$

$$\Rightarrow \quad y_2 X = 0$$

$$X = 0 \qquad \because y_2 \neq 0$$

Thus $G = 0 \qquad X = 0 = Y$

$\Rightarrow$ The forces are in equilibrium.

EXERCISE : 2

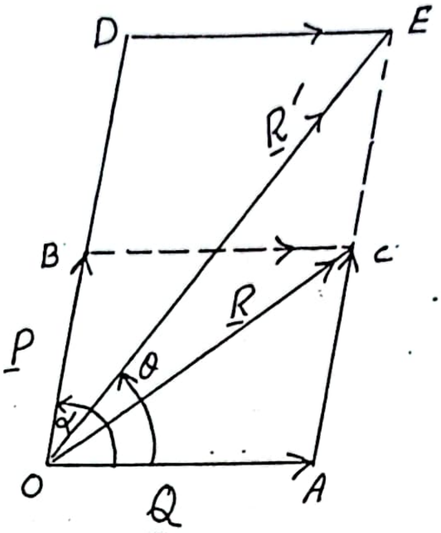
Q: 1 # If two forces P and Q act at such an angle that their resultant $R = P$, show that if P is doubled the new resultant is at right angles to force Q.

Sol # Let the forces P, Q act at an angle α and R is their resultant. Then

$$R^2 = P^2+Q^2+2PQ\cos\alpha$$

But it is given that $R=P$

$$\Rightarrow P^2 = P^2+Q^2+2PQ\cos\alpha$$



$$\Rightarrow Q^2+2PQ\cos\alpha = 0$$

$$\Rightarrow Q(Q+2P\cos\alpha)=0$$

$$\Rightarrow Q+2P\cos\alpha = 0 \quad (\because Q\neq 0)$$

$$\Rightarrow \cos\alpha = -\frac{Q}{2P} \longrightarrow ①$$

Let $\underline{R}'$ be the new resultant when $\underline{P}$ is doubled.

Let $\underline{R}'$ makes angle θ with Q, then

$$\tan\theta = \frac{2P\sin\alpha}{Q+2P\cos\alpha} \qquad \left(\because \tan\theta = \frac{F_2\sin\alpha}{F_1+F_2\cos\alpha}\right)$$

using ①

$$\tan\theta = \frac{2P\sin\alpha}{0} = \infty$$

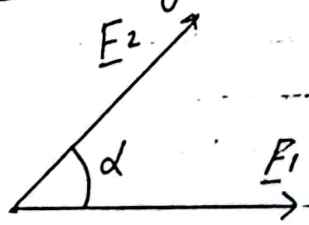
$$\theta = \frac{\pi}{2}$$

Thus the new resultant is at right angle to Q

Q:2 # The greatest resultant that two forces can have is of magnitude P and the least is of magnitude Q. Show that, when they act at an angle α, their resultant is magnitude $\sqrt{P^2\cos^2\left(\frac{\alpha}{2}\right) + Q^2\sin^2\frac{\alpha}{2}}$.

Sol # Let the forces be $\underline{F_1}$ and $\underline{F_2}$ acting at an angle α. Suppose that

$$F_1 > F_2$$



The max resultant is given by

$$R_{max} = F_1 + F_2$$

But given that $R_{max} = P$

$$\Rightarrow P = F_1 + F_2 \longrightarrow ①$$

The min resultant is

$$R_{min} = F_1 - F_2$$

But given that $R_{min} = Q$

$$\Rightarrow Q = F_1 - F_2 \longrightarrow ②$$

squaring & Adding ① & ②

$$P^2 + Q^2 = 2(F_1^2 + F_2^2)$$

$$\Rightarrow F_1^2 + F_2^2 = \frac{P^2+Q^2}{2} \longrightarrow ③$$

Squaring ① & ② and subtracting

$$P^2 - Q^2 = 4F_1F_2$$

$$\Rightarrow F_1F_2 = \frac{P^2-Q^2}{4} \longrightarrow ④$$

Magnitude of the resultant is

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha}$$

$$= \sqrt{\frac{P^2+Q^2}{2} + 2\left(\frac{P^2-Q^2}{4}\right)\cos\alpha} \quad \text{using ③ \& ④}$$

$$= \sqrt{\frac{P^2+Q^2}{2} + \left(\frac{P^2-Q^2}{2}\right)\cos\alpha}$$

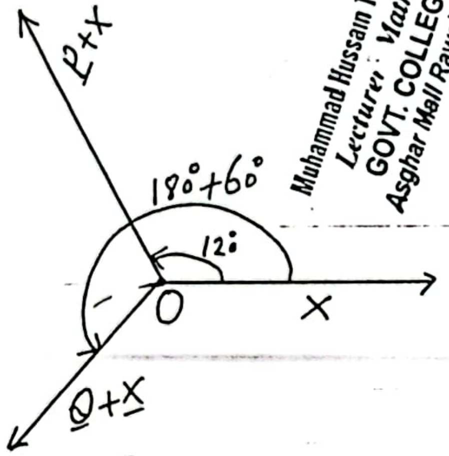
$$= \sqrt{\frac{P^2}{2}(1+\cos\alpha) + \frac{Q^2}{2}(1-\cos\alpha)}$$

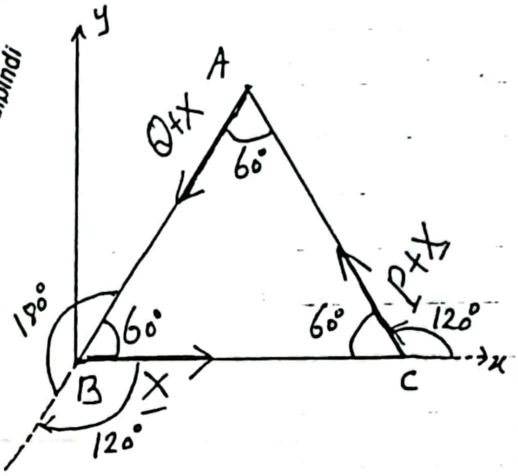
$$= \sqrt{\frac{P^2}{2}\cdot 2\cos^2\alpha/2 + \frac{Q^2}{2}\cdot 2\sin^2\alpha/2}$$

$$= \sqrt{P^2\cos^2\alpha/2 + Q\sin^2\alpha/2}$$

Q: 3 # Forces $\underline{X}$, $\underline{P}+\underline{X}$, $\underline{Q}+\underline{X}$ act at a point in the directions of the sides of an equilateral triangle, taken one way round. Show that they are equivalent to two forces P and Q acting at an angle. 120°

Sol # **Equilateral Triangle #** A triangle whose all sides are of equal length. Its all angles will also be equal





Let the side BC of ΔABC be along x-axis and forces $\underline{X}$, $\underline{P}+\underline{X}$, $\underline{Q}+\underline{X}$ act at a point O along side BC, CA, AB of triangle. Let X′, Y′ be the resolved parts of the resultant. Then

$$X' = X\cos 0^\circ + (P+X)\cos 120^\circ + (Q+X)\cos(180^\circ+60^\circ)$$

$$= X + (P+X)\cos(90^\circ+30^\circ) - (Q+X)\cos 60^\circ$$

$$= X - \tfrac{1}{2}(P+X) - \tfrac{1}{2}(Q+X)$$

$$= -\frac{P+Q}{2}$$

$$Y' = X\sin 0^\circ + (P+X)\sin(90^\circ+30^\circ) + (Q+X)\sin(180^\circ+60^\circ)$$

$$= 0 + \frac{\sqrt{3}}{2}(P+X) - \frac{\sqrt{3}}{2}(Q+X) = \frac{\sqrt{3}}{2}(P-Q)$$

$$R = \sqrt{X'^2 + Y'^2}$$

$$= \sqrt{\frac{(P+Q)^2}{4} + \frac{3}{4}(P-Q)^2}$$

$$= \sqrt{\frac{1}{4}(P^2+Q^2+2PQ+3P^2+3Q^2-6PQ)}$$

$$= \sqrt{P^2+Q^2 \cancel{+} PQ}$$

$$= \sqrt{P^2+Q^2+2PQ(-1/2)}$$

Let α be angle between forces P & Q. Then.

$\cos\alpha = -1/2$.

Reference angle $\theta_r = 60°$

α lies in IInd and IIIrd quadrants.

IInd	IIIrd
$\alpha = 180° - \theta_r$	$\alpha = 180° + \theta_r$
$= 180° - 60°$	$= 180° + 60° = 240°$

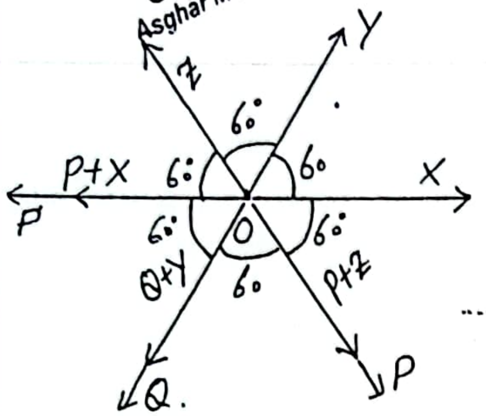
But angle between two lines can not be greater than 180°.

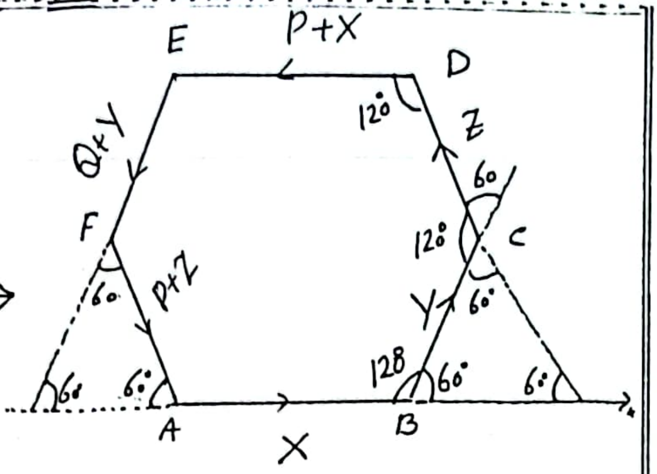
Therefore $\alpha = 120°$

Thus forces are equivalent to two forces P & Q acting at angle 120°

Q:4 # Forces $X, Y, Z, P+X, Q+Y, R+Z$ act at a point in the directions of sides of regular hexagon taken one way round. Show that their resultant is a force $P+Q$ in the direction of the force $Q+Y$.

Sol # Let the side AB of hexagon be along x-axis and force act in the direction of AB. The other forces acting at point O in the directions of the sides of hexagon are shown


Y
Z
60°
P+X
60°
60
X
P
60°
O
60°
Q+Y
60
P+Z
Q
P


E
P+X
D
120°
Z
Q+Y
60
F
120°
C
P+Z
60
Y
60°
60
60°
128
60°
60°
A
X
B

I- Method # (a) Force P+X , X are opposite and their resultant is min of magnitude P along P+X .

(b) Forces Y , Q+Y have min resultant Q along Q+Y

(c) Forces P , P+Z have min resultant P along P+Z .

Hence the system reduces to three forces P, P, Q and Q bisects angle between P, P.

Now the resultant two equal forces $P, P = 2P\cos\left(\frac{120°}{2}\right)$

$= 2P\cos 60° = 2P \cdot \frac{1}{2} = P$

and it bisects the angle between P, P i.e is along Q+Y

Now we have two forces P, Q along Q+Y and theire resultant P+Q is also along Q+Y.

OR

If X' Y' are the resolved parts of the resultant, then

$$X' = P\cos 180° + Q\cos(180°+60°) + P\cos(-60°)$$

$$= -P - \frac{1}{2}Q + \frac{1}{2}P = -\frac{1}{2}(P+Q)$$

$$Y' = P\sin 180° + Q\sin(180°+60°) + P\sin(-60°)$$

$$= 0 - \frac{\sqrt{3}}{2}Q - \frac{\sqrt{3}}{2}P$$

$$= -\frac{\sqrt{3}}{2}(P+Q)$$

Required resultant is

$$R = \sqrt{X'^2 + Y'^2}$$

$$= \sqrt{\frac{1}{4}(P+Q)^2 + \frac{3}{4}(P+Q)^2}$$

$$= P+Q.$$

If this resultant makes angle θ with x-axis, we have

$$\tan\theta = \frac{Y'}{X'} = \frac{-\frac{\sqrt{3}}{2}(P+Q)}{-\frac{1}{2}(P+Q)}$$

$$= \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$$

and θ lies in IIIrd quadrant because both X', Y' are -ve. Hence

$$\theta = 180^\circ + 60^\circ = 240^\circ$$

But the angle of $Q+Y$ with x-axis is 240°

Therefore resultant $P+Q$ is in the direction of $Q+Y$.

OR

Resolving the forces along x-axis and y-axis

$$X' = X\cos 0^\circ + Y\cos 60^\circ + Z\cos 120^\circ + (P+X)\cos 180^\circ + (Q+Y)\cos(180^\circ+60^\circ) + (P+Z)\cos(-60^\circ)$$

$$= X + \frac{1}{2}Y - \frac{1}{2}Z - (P+X) - \frac{1}{2}(Q+Y) + \frac{1}{2}(P+Z) = -\frac{1}{2}(P+Q)$$

$$Y' = X\sin 0^\circ + Y\sin 60^\circ + Z\sin 120^\circ + (P+X)\sin(-60^\circ) + (Q+Y)\sin(180^\circ+60^\circ) + (P+Z)\sin(-60^\circ)$$

$$= -\frac{\sqrt{3}}{2}(P+Q)$$

$$R = P+Q$$

$$\tan\theta = \sqrt{3} \Rightarrow \theta = 240^\circ$$

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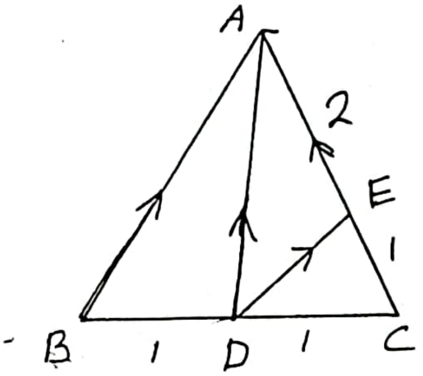
Q:5 # Forces $2\overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{BA}$ act along the sides of a triangle ABC. Show that their resultant is $6\overrightarrow{DE}$, Where D bisects BC and E is a point on CA such that $CE = \frac{1}{3}CA$

Sol #



By (λ, u) Theorem

$$\overrightarrow{BA} + \overrightarrow{CA} = (1+1)\overrightarrow{DA}$$

$$= 2\overrightarrow{DA}$$

where D bisects BC.

Now $\overrightarrow{BC} = 2\overrightarrow{DC}$

$$\Rightarrow 2\overrightarrow{BC} = 4\overrightarrow{DC}$$

So given three forces are equivalent to (three) forces $4\overrightarrow{DC}$ and $2\overrightarrow{DA}$.

By (λ, μ) Theorem

$$\text{Resultant} = \overrightarrow{CA} + \overrightarrow{BA} + 2\overrightarrow{BC}$$

$$= 2\overrightarrow{DA} + 4\overrightarrow{DC}$$

$$= 2(1\cdot\overrightarrow{DA} + 2\overrightarrow{DC})$$

$$= 2\cdot(1+2)\overrightarrow{DE} = 6\overrightarrow{DE}$$

where E divides CA in ratio 1:2.

$$\Rightarrow CE = \tfrac{1}{3}CA$$

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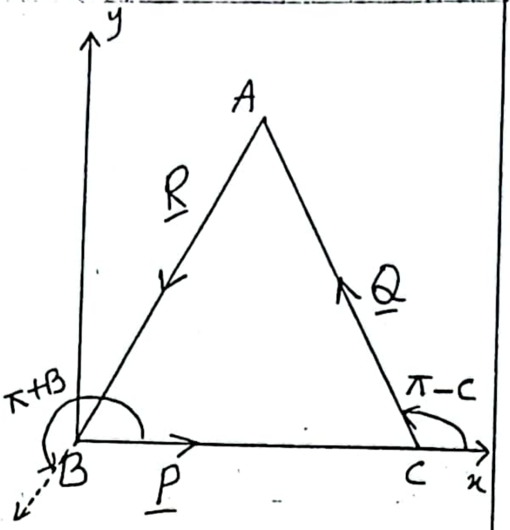
Q:6 # Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC. Find the Condition that their resultant is parallel to BC and determine its magnitude

Sol # Since the forces act along the sides of triangle, they are co-planar. Let the side BC of triangle be along x-axis and y-axis ⊥ar to it

Resolving the forces along & $\perp$ar to x-axis, we have



$$X = P\cos 0^\circ + Q\cos(\pi - C) + R\cos(\pi + B)$$

$$= P - Q\cos C - R\cos B$$

$$Y = P\sin 0^\circ + Q\sin(\pi - C) + R\sin(\pi + B)$$

$$= 0 + Q\sin C - R\sin B$$

The resultant will be parallel to BC (i.e x-axis) if

$$Y = 0$$

if $\quad Q\sin C - R\sin B = 0$

if $\quad Q\sin C = R\sin B$

if $\quad \dfrac{\sin B}{\sin C} = \dfrac{Q}{R}$

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By Law of Sines

$$\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \frac{\sin B}{\sin C} = b/c$$

$$\Rightarrow \quad b/c = \frac{Q}{R}$$

$$\Rightarrow \quad R = \frac{Qc}{b} \longrightarrow ① \text{ Condition}$$

Magnitude of the resultant is

$$R' = \sqrt{X^2+Y^2} = \sqrt{X^2+0} = X$$

$$= P - Q\cos C - R\cos B$$

$$= P - \left(Q\cos C + \frac{Qc}{b}\cos B\right)$$

$$= P - Q\left(\frac{b\cos C + c\cos B}{b}\right)$$

$$= P - Q\frac{a}{b} \quad \text{(By Projection formula)}$$

Remarks # Forces are not completely represent by the sides of triangle otherwise they were be equal to a couple and their resultant will be zero because if the forces are completely represented, in magnitude, direction and position by the sides a closed plane polygon, taken one way round, they are equivalent to a couple whose moment is twice the area of the polygon.

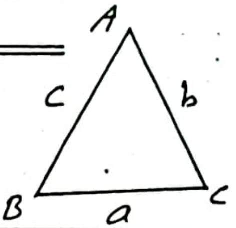
Revision

Projection Formula

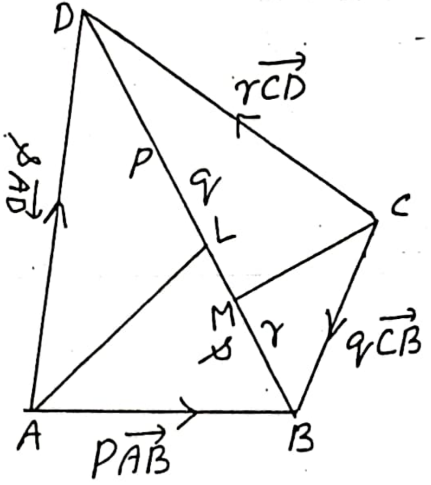
(a) $a = b\cos C + c\cos B$

(b) $b = a\cos C + c\cos A$

(c) $c = a\cos B + b\cos A$



Q:7 # If forces $p\vec{AB}, q\vec{CB}, r\vec{CD}, s\vec{AD}$ acting the sides of a plane quadrilateral are in equilibrium, show that $pr = qs$



Sol # The forces are as shown.

By $(\lambda-\mu)$ Theorem, we have

$p\vec{AB} + s\vec{AD} = (p+s)\vec{AL} \rightarrow ①$

Where L is a point on BD such that

$$\frac{BL}{LD} = \frac{s}{p} \rightarrow ②$$

and $q\vec{CB} + r\vec{CD} = (q+r)\vec{CM} \rightarrow ③$

Where M is a point on BD such that

$$\frac{BM}{MD} = \frac{r}{p} \rightarrow ④$$

Thus the given system of four forces in equilibrium.

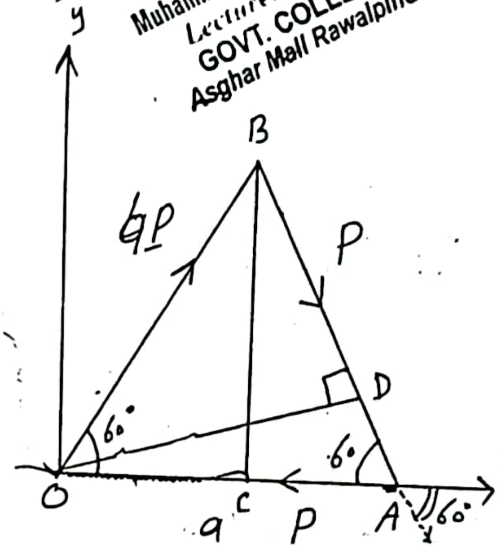
is equivalent to two forces $(p+s)\overrightarrow{AL}$ and $(q+r)\overrightarrow{CM}$. These two forces must balance each other and hence must be collinear, therefore L & M are coincident and we have from equations ② & ④

$$\frac{s}{p} = \frac{r}{q} \Rightarrow pr = qs$$

Q:8 # (Forces) OAB is an equilatoral triangle of side (a) : C is mid point of OA. Forces 4P and p act along the sides OB, BA and AO respectively. If OA and OY (||al to BC) are taken as x- and y-axes, prove that the resultant of the forces is 3P and the equation of the line of action of the resultant is

$$3y = \sqrt{3}(3x+a)$$

Sol # The forces are as shown. Resolving along and perpendicular to the x-axis, we get



$OD = a \sin 60°$

$$X = P\cos 180° + P\cos(-60°) + 4P\cos 60°$$

$$= -P + \tfrac{1}{2}P + 4P(\tfrac{1}{2})$$

$$= -P + \tfrac{1}{2}P + 2P$$

$$= \tfrac{3}{2}P$$

$$Y = P\sin 180° + P\sin(-60°) + 4P\sin 60°$$

$$= 0 - \frac{\sqrt{3}}{2}P + 4P\frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2}(-P+4P) = \frac{3\sqrt{3}}{2}P$$

Resultant is

$$R = \sqrt{X^2+Y^2} = \sqrt{\left(\tfrac{3}{2}P\right)^2 + \left(\tfrac{3\sqrt{3}P}{2}\right)^2}$$

$$R = \sqrt{\left(\frac{3}{2}P\right)^2 + \left(\frac{3\sqrt{3}P}{2}\right)^2}$$

$$= \sqrt{\frac{9}{4}P^2 + \frac{27P^2}{4}} = \sqrt{\frac{36P^2}{4}} = 3P$$

Taking moment about O

$G =$ The sum of moments of all the forces about O.

$$= -P\,(a\sin 60^\circ) = -\frac{\sqrt{3}}{2}aP$$

Hence equation of line of action of the resultant is

$$G - xY + yX = 0.$$

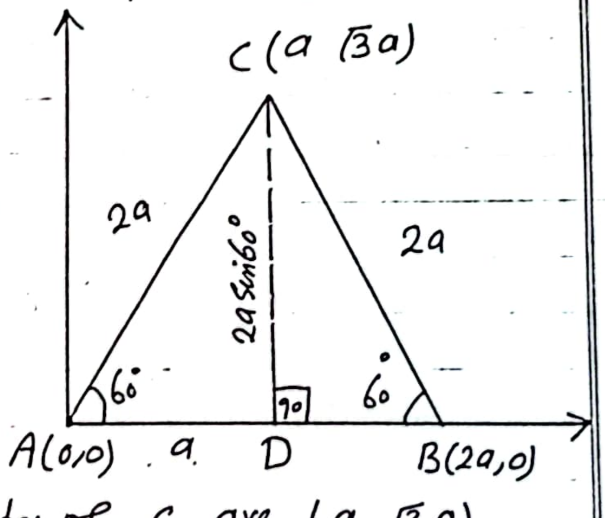
$$-\frac{\sqrt{3}}{2}aP - \frac{3\sqrt{3}}{2}xP + \frac{3}{2}yP = 0$$

$$-\sqrt{3}\,aP - 3\sqrt{3}\,Px + 3yP = 0$$

$$3y = \sqrt{3}\,(a + 3x)$$

Q:10 # A system of forces acts on a plate in the form of an equilateral triangle of side $2a$. The moments of the forces about the three vertices are G_1, G_2, G_3 respectively. Find the magnitude of the resultant

Sol # Let ΔABC be equilateral triangle bound the the plate. Let side AB be taken along x-axis. Join C to mid point D of side AB. Then CD will be perpendicular to AB.



The Co-ordinates of C are $(a, \sqrt{3}a)$.
Let X, Y be the resolved parts of resultant and G_1, G_2, G_3 be the sum of moments about points A, B, C respectively

The sum of moments about A(0,0) $= G - 0Y + 0X$

$$G_1 = G \longrightarrow ①$$

The sum of moments about B(2a,0)

$$= G_2 = G - 2aY + 0X$$

$$G_2 = G_1 - 2aY$$

$$Y = \frac{G_1 - G_2}{2a} \longrightarrow ②$$

The sum of moments about C(a $\sqrt{3}a$)

$$= G - aY + \sqrt{3}aX$$

$$G_3 = G_1 - aY + \sqrt{3}aX$$

putting $Y = \frac{G_1 - G_2}{2a}$

$$G_3 = G_1 - a\left(\frac{G_1 - G_2}{2a}\right) + \sqrt{3}.aX$$

$$= G_1 - \frac{G_1}{2} + \frac{G_2}{2} + \sqrt{3}aX$$

$$2G_3 = G_1 + G_2 + \sqrt{3}aX$$

$$X = \frac{2G_3 - G_1 - G_2}{2\sqrt{3}a}$$

The resultant R is given by

$$R = \sqrt{X^2 + Y^2}$$

$$= \sqrt{\left(\frac{2G_3 - G_1 - G_2}{2\sqrt{3}a}\right)^2 + \left(\frac{G_1 - G_2}{2a}\right)^2}$$

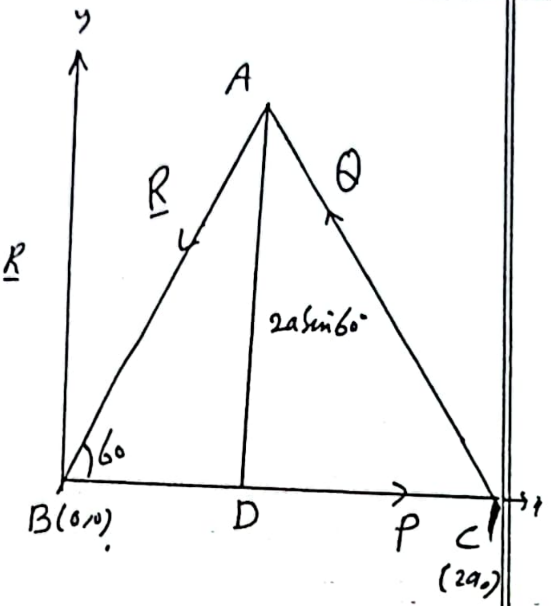
$$R^2 = \frac{1}{12a^2}\left[4G_3^2 + G_1^2 + G_2^2 - 4G_1G_3 - 4G_2G_3 + 2G_1G_2 + 3G_1^2 + 3G_2^2 - 6G_1G_2\right]$$

$$= \frac{1}{12a^2}\left[4G_1^2 + 4G_2^2 + 4G_3^2 - 4G_1G_2 - 4G_1G_3 - 4G_2G_3\right]$$

$$= \frac{1}{3a^2}\left[G_1^2 + G_2^2 + G_3^2 - G_1G_2 - G_1G_3 - G_2G_3\right]$$

$$R = \sqrt{\frac{G_1^2 + G_2^2 + G_3^2 - G_1G_2 - G_1G_3 - G_2G_3}{3a^2}}$$

2nd Method



Let ΔABC be the equilateral plate of side $2a$.

Let the forces be P, Q, R acting on the plate along the sides of ΔABC.

Since G_1, G_2, G_3 are moments about the vertices of ΔABC.

$$G_1 = P \cdot 2a \sin 60^\circ$$

$$= P \cdot 2a \frac{\sqrt{3}}{2} = Pa\sqrt{3}$$

$$P = \frac{G_1}{\sqrt{3}a} \longrightarrow ①$$

$$Q = \frac{G_2}{\sqrt{3}a} \longrightarrow ②$$

$$R = \frac{G_3}{\sqrt{3}a} \longrightarrow ③$$

Resolving the forces along and perpendicular to BC

$$X = P\cos 0^\circ + Q\cos 120^\circ + R\cos(180^\circ+60^\circ)$$

$$= P - Q\sin 30^\circ - R\cos 60^\circ$$

$$= P - \frac{1}{2}Q - \frac{1}{2}R$$

$$= \frac{G_1}{\sqrt{3}a} - \frac{G_2}{2\sqrt{3}a} - \frac{G_3}{2\sqrt{3}a}$$

$$= \frac{1}{2\sqrt{3}a}(2G_1 - G_2 - G_3)$$

$$Y = P\sin 0^\circ + Q\sin 120^\circ + R\sin(180^\circ+60^\circ)$$

$$= Q\cos 30^\circ - R\sin 60^\circ$$

$$= \frac{\sqrt{3}}{2}Q - \frac{\sqrt{3}}{2}R$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{G_2}{\sqrt{3}a} - \frac{\sqrt{3}}{2}\cdot\frac{G_3}{\sqrt{3}a} = \frac{G_2 - G_3}{2a},$$

$R = \sqrt{x^2+y^2}$

Revision # (From Geometry)

. **Median of Triangle #** A median is the line segment joining a vertex of triangle to the middle point of the opposite vertex

* **Isosceles Triangle #** A Triangle having two sides equal

* **Equilateral Triangle** A Triangle having all sides equal.

* The medians of the Congruent sides of of an isosceles triangle are Congruent

* Three medians of an equilateral triangle are Congruent.

* The median bisecting the base of an isosceles triangle bisects the vertex angle and is perpendicular to the base.
(This concept is used in above question)

* The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

* If the bisector of the vertex of an isosceles triangle bisects the base, then triangle is ~~Con~~ isosceles

Altitudes of Triangle

The line segments which are perpendiculars from the vertices to the opposite sides of a triangle are calledits altitude.

* The altitudes of the Congruents sides of an isosceles triangle are Congurent

* If two altitudes of triangle are Conguent, the triangle is isosceles.

* If all the three altitudes of a triangle are Conguent, the triangle is equilateral.

* Three medians of an equilator triangle are congruent.

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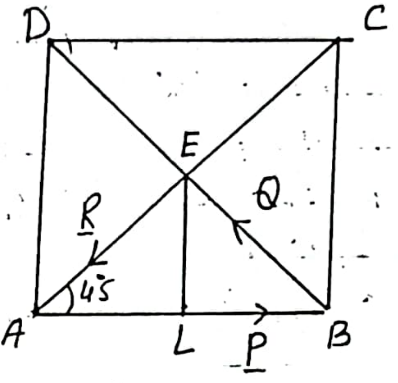
Q: 11# A Couple of moment G acts on a square board $ABCD$ of side a. Replace the couple by forces acting along AB, BD and CA.



Sol Let P, Q & R be the required forces acting along sides AB BD & CA respectively

$AE = BE = \frac{AC}{2}$

$AB = BC = a$

$\therefore EL = \frac{a}{2}$ ($\perp$ar from the vertex from the vertex of an isosceles triangle bisects the base and is half of its length).

Since the forces are equivalent to couple G, therefore the sum of (all) moments of all the forces about any point must be same.

Taking moment about A

$$G = Q|AE| = Q\frac{|EL|}{\sin 45^\circ}$$

$\frac{EL}{AE} = \sin 45^\circ$

$$= Q \cdot \frac{a/2}{\frac{1}{\sqrt{2}}} = \frac{Qa}{\sqrt{2}}$$

$$\Rightarrow Q = \frac{\sqrt{2}G}{a}$$

Taking moment about B

$$G = R|BE| = R \cdot \frac{a}{\sqrt{2}}$$

$$\Rightarrow R = \frac{\sqrt{2}G}{a}$$

Taking moment about E, we get

$$G = P|LE| = P \cdot \frac{a}{2}$$

$$P = \frac{2G}{a}$$

Hence couple is replaced by forces $P = \frac{2G}{a}$ along AB, $Q = \frac{\sqrt{2}G}{a}$ along BD and $R = \frac{\sqrt{2}G}{a}$ along CA

2nd Method

Resolving the forces along and perpendicular to AB

$$X = P\cos 0^\circ + Q\cos(90^\circ+45^\circ) + R\cos(180^\circ+45^\circ)$$
$$= P - \frac{1}{\sqrt{2}}Q - \frac{1}{\sqrt{2}}R$$

$$Y = P\sin 0^\circ + Q\sin(90^\circ+45^\circ) + R\sin(180^\circ+45^\circ)$$
$$= 0 + \frac{1}{\sqrt{2}}Q - \frac{1}{\sqrt{2}}R$$

$\because$ net effect of forces $\underline{P}, \underline{Q}, \underline{R}$ must be a Couple of moment G

$X = 0 \qquad\qquad Y = 0$

$$\Rightarrow P - \frac{1}{\sqrt{2}}Q - \frac{1}{\sqrt{2}}R = 0 \quad \rightarrow ①$$

$$\& \quad \frac{1}{\sqrt{2}}Q - \frac{1}{\sqrt{2}}R = 0$$

$$Q - R = 0$$
$$Q = R$$

putting $Q = R$ in ①

$$P - \frac{Q}{\sqrt{2}} - \frac{Q}{\sqrt{2}} = 0 \Rightarrow P - \sqrt{2}Q = 0$$
$$\Rightarrow P = \sqrt{2}Q$$

Hence forces are $\sqrt{2}Q, Q, Q$

Taking moment about A

Sum of moment of forces $= G = Qa\sin 45^\circ$

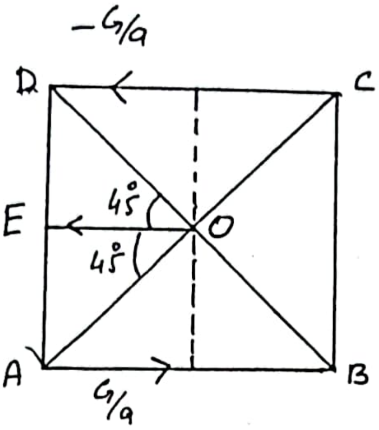
$$= \frac{Qa}{\sqrt{2}}$$

$$\Rightarrow Q = \frac{\sqrt{2}G}{a}$$

Thus forces are

$$P = 2G/a$$
$$Q = \sqrt{2}G/a$$
$$R = Q = \frac{\sqrt{2}G}{a}$$

3rd Method



Let the forces of the Couple of moment G acting along sides AB & CD with Couple arm a.

$$G = \text{Force} \times \text{Couple arm}$$
$$= \text{Force} \times a$$
$$\text{Force} = G/a$$

$\Rightarrow$ Couple is $(G/a \quad -G/a)$

We replace the couple into an equivalent Couple $(-2G/a \quad 2G/a)$ such that $-2G/a$ acts along OE and $2G/a$ acting along AB. Resolving along BD and CA the force $-2G/a$. We have.

$$\text{Force along } BD = \frac{2G}{a}\cos 45^\circ$$
$$= \frac{2G}{a}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{2}G}{a}$$

$$\text{Force along } CA = \frac{2G}{a}\cos 45^\circ = \frac{\sqrt{2}G}{a}$$

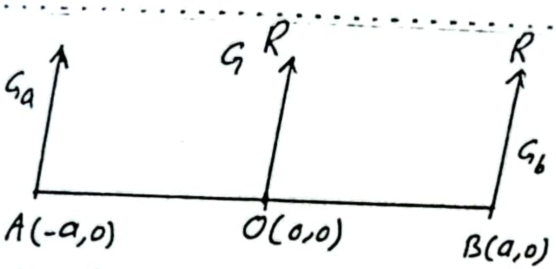
Hence Couple G is replaced by forces $\frac{2G}{a}, \frac{\sqrt{2}G}{a}, \frac{\sqrt{2}G}{a}$ acting along AB, BD, CA.

Q:12 # A, B are any two points in a lamina, on which a system of forces Coplanar with it are acting and when the forces are reduced to a single force at each of these points and a Couple, the moments of the Couples are G_a, G_b respectively. Prove that when the reduction is made to a force at mid. point of AB and a Couple, the moments of the Couple is $\frac{1}{2}(G_a + G_b)$

Sol # Lamina: A flat thin body in the form of a plate or sheet of uniform thickness and of Constant density.

A thin flat plate or sheet of uniform thickness and of Constant density.

Let $AB = 2a$ and Centre of AB is taken as origin. Then Co-ordinates of A and B are $(-a,0)$, $B(a,0)$



G_a
G
R
R
G_b
$A(-a,0)$
$O(0,0)$
$B(a,0)$

Let R be the force and G be the moment of couple when reduction is made at origin.

When the reduction is made at point A

$$G_a = G + aR \longrightarrow ①$$

When the reduction is made at point B

$$G_b = G_a - aR \longrightarrow ②$$

OR

Let X, Y be the resolved parts of the Resultant.
When the reduction is made at (origin) A

$$G_a = G - (-a)Y + 0 \cdot X$$
$$G_a = G + aY \longrightarrow ①$$

When the reduction is made at point $B(a,0)$

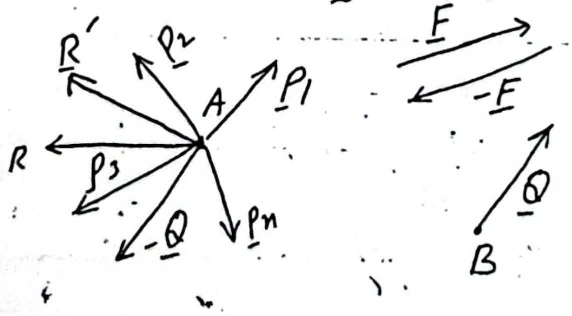
$$G_b = G - aY + 0X$$
$$= G - aY \longrightarrow ②$$

Adding ① & ②

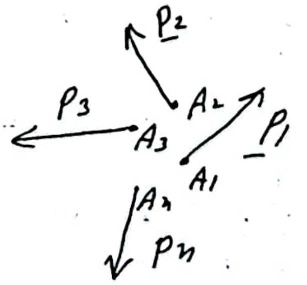
$$2G = G_a + G_b$$
$$G = \frac{G_a + G_b}{2}$$

Q:13 # Show that a system of Coplanar forces may be represented in an infinite no of ways by two forces, one through each of two fixed points in their plane.

Sol # Let $\underline{P}_i$ $(i = 1, 2, \dots n)$ be n Coplanar forces and A, B be any two fixed points in the plane of forces



$\underline{R}'$
$\underline{P}_2$
A
$\underline{P}_1$
$\underline{F}$
$-\underline{F}$
R
$\underline{P}_3$
$-\underline{Q}$
$\underline{P}_n$
$\underline{Q}$
B



$\underline{P}_2$
P_3
A_2
A_3
$\underline{P}_1$
A_1
A_n
P_n

Now when the reduction of forces is made at A, we have a single force

$$\underline{R} = \underline{P_1} + \underline{P_2} + \underline{P_3} + \dots + \underline{P_n} \rightarrow ①$$

and n couples $\underline{G_i}$ which be equivalent to a single (force) Couple $(\underline{F}, -\underline{F})$ of moment $\underline{G}$ s. that

$$\underline{G} = \underline{G_1} + \underline{G_2} + \underline{G_3} + \dots + \underline{G_n}$$

Replace the couple $(\underline{F}, -\underline{F})$ be an equivalent Couple $(\underline{Q}\ -\underline{Q})$ of the same moment lying in the same plane such that $\underline{Q}$ acts B and $-\underline{Q}$ at A. At point A we have single resultant force $\underline{R}'$ given by

$$\underline{R}' = \underline{R} + (-\underline{Q})$$

and at B there will be a single force $\underline{Q}$

As such we have reduced the system into two forces one through each of fixed points A & B.

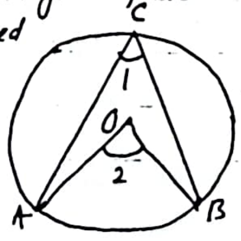
Now since the couple $(\underline{F}\ -\underline{F})$ can be replaced by an equivalent couple in an infinite number of ways, hence the above reduction can be done in an infinite number of ways.

Concepts From Geometry to be used in next Question

★ **Circumcentre** # It is the point of concurrency of perpendicular bisectors of the sides of triangle.

★ The central angle of any minor arc of a circle, is double (in measure) of the inscribed angle of the corresponding major arc

★ An inscribed angle of an arc is the angle whose vertex is at any point of the arc and whose arms pass through ends of the arc

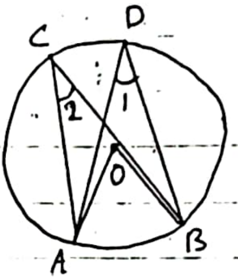


∠ACB is inscribed angle of major arc AB and ∠AOB is central angle of minor arc AB

$m\angle 2 = 2m\angle 1$

★ All angles inscribed in a major arc are are equal in measure

$m\angle 1 = m\angle 2$



★ Angle inscribed in a major arc is acute. This follows from the fact that

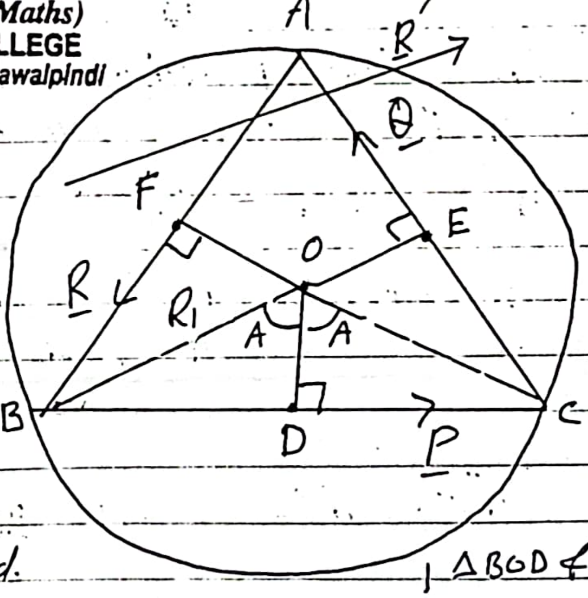
$m\angle AOB < 180^\circ$

★ The perpendicular from the vertex of an isosceles triangle at the mid point of the base biseds the vertex angle.

Q:14# Three forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC prove that if $P\cos A + Q\cos B + R\cos C = 0$, then the line of action of the resultant passes through the circumcentre of the triangle.

Sol #

Let ABC be the triangle and ⊥ar bisectors OD, OE, OF of sides meet at point O. Then O is circumcentre of ΔABC.



Now angle at centre of the minor arc BC is double the inscribed angle of the corresponding major arc.

ΔBOD & ΔCOD are Congruent

$\therefore m\angle BOC = 2m\angle A$

∴ Again ΔOBC is an isosceles triangle because $OB = OC$. Since ⊥ar from the vertex of an isosceles triangle at the mid point of base bisects the vertex, therefore

$m\angle BOD = m\angle A = m\angle COD$.

Let R_1 be the radius of circumcircle.

In $\triangle BOD$ $\quad OD = R_1 \cos A$

In $\triangle AOC$ $\quad OE = R_1 \cos B$

In $\triangle AOB$ $\quad OF = R_1 \cos C$

Let d be the $\perp$ar distance of the resultant R' from O. Taking moment about O

$$R'd = P\cdot OD + Q\cdot OE + R\cdot OF$$
$$= PR_1\cos A + Q\, R_1\cos B + R R_1 \cos C$$
$$= R_1(P\cos A + Q\cos B + R\cos C)$$

If $P\cos A + Q\cos B + R\cos C = 0$

Then

$$R'd = 0$$
$$\Rightarrow d = 0 \quad \because R' \neq 0$$

$\Rightarrow$ Resultant passes through O

OR

The line of action of resultant will pass through O if the sum of moments about $O = 0$

If $\quad P\cdot OD + Q\cdot OE + R\cdot OF = 0$

If $\quad PR_1\cos A + QR_1\cos B + RR_1\cos C = 0$

If $\quad R_1(P\cos A + Q\cos B + R\cos C) = 0$

If $\quad P\cos A + Q\cos B + R\cos C = 0$

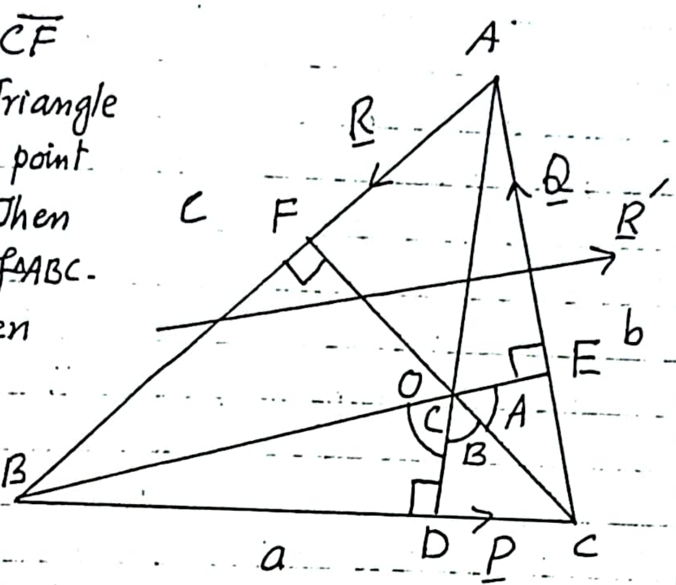
Orthocentre of Triangle

It is the point of concurrency of altitudes.

Q: 15 # Three forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC. Prove that if

$$P\sec A + Q\sec B + R\sec C = 0,$$ Then the line of action of the resultant passes through The

the orthocentre of the triangle.



Sol # Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be the altitudes of Triangle ABC and O be the point of their concurrency. Then O is the orthocentre of $\triangle ABC$. Since the angle between the lines is the same as the angle between their perpendiculars, we have

$\angle BOF = \angle COE = \angle A$ etc

From $\triangle BDO$, we have

$$\frac{OD}{BD} = \cot C$$

$$OD = BD\cot C \rightarrow ①$$

From $\triangle BAD$

$$\frac{BD}{c} = \cos B$$

$$BD = c\cos B$$

using in ①

$$OD = c\cos B\cot C$$

Similarly

$$OE = a\cos C\cot A$$

$$OF = b\cos A\cot B$$

Let R' be the resultant at distance d from O, then by Varignon's theorem

$$R'd = P\,DO + Q\,OE + R\,OF$$

$$= P\,c\cos B\cot C + Q\,a\cos C\cot A + R\,b\cos A\cot B$$

$$= P\,c\cos B\frac{\cos C}{\sin C} + Q\,a\cos C\frac{\cos A}{\sin A} + R\,b\cos A\frac{\cos B}{\sin B}$$

By Law of Sines

$$\frac{c}{\sin C} = \frac{a}{\sin A} = \frac{b}{\sin B} = k$$

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$R d' = k\cos B\cos C\cdot P + Q k\cos C\cos A + k R\cos A\cos B$

$= k\cos B\cos C\cos A\left[\frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C}\right]$

$= k\cos B\cos C\cos A\,[P\sec A + Q\sec B + R\sec C]$

If $P\sec A + Q\sec B + R\sec C = 0$

$R d' = 0$

$\Rightarrow d' = 0 \quad \because R' \neq 0$

$\Rightarrow$ The resultant passes through O

OR

The line of action of the resultant will pass through O

If Sum of moments about O $= 0$

If $P c\cos B\frac{\cos C}{\sin C} + Q a\cos C\frac{\cos A}{\sin A} + R b\cos A\frac{\cos B}{\sin B} = 0$

If $P\cos B\cos C\left(\frac{c}{\sin C}\right) + Q a\cos C\left(\frac{a}{\sin A}\right)\cos A + R\cos A\cos B\left(\frac{b}{\sin B}\right) = 0$

If $P\cos B\cos C\cdot k + Q\cos A\cos C k + R\cos A\cos B\cdot k = 0$

If $P\cos B\cos C + Q\cos A\cos C + R\cos A\cos B = 0$

Dividing by $\cos A\cos B\cos C$

If $\frac{P}{\cos A} + \frac{Q}{\cos B} + \frac{R}{\cos C} = 0$

if $P\sec A + Q\sec B + R\sec C = 0$

Geometry to be Used Next

Similar Geometric Figures#

Two geometric figures are similar if one can be made conguent to the other by using a transformation of similitude i.e if one is a magnification or the reduction of the other.

* If Corresponding lengths in two similar figures are in ratio k, then the corresponding areas are in ratio k^2 and the corresponding volumes are in ratio k^3.

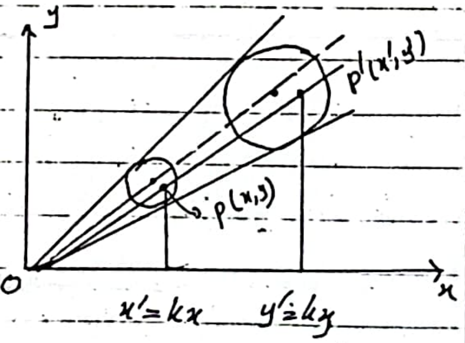
Transformation of Similitude

The transformation $x' = kx$, $y' = ky$, in rectangular Co-ordinates. It multiplies the distance between every two points by the same Constant k, called the ratio of similitude. If k is less than 1, the transformation is said to shrink the plane. This transformation is also called the homothetic transformation.

Radially Related Figures

Figures which are central projections of each other; figures such that a line drawn from some fixed point to a point of one of them passes through a point of the other, such that the ratio of the distances from the two fixed points is always same (Two similar figures can always be so placed). The fixed point is the homothetic centre, the centre of similitude or ray centre. The ratio of the two line segments is the ray ratio, ratio of similitude or homothetic ratio. The two radially related figures are similar. They are also called homothetic figures.

In fig, the circumference of the larger circle k times the circumference of the smaller, and the point $P(x', y')$ is k times as far from the origin (fixed point) as the point P. This is the homothetic transformation.



Similar Triangles

Two triangles are similar iff corresponding angles are equal. The corresponding sides are then proportional

For Triangles to be Congruent

- ✯ If A·S·A ≅ A·S·A, then triangles are congruent
- ✯ S·A·A ≅ S·A·A i.e in correspondence one side and its opposite angle and another of the triangle is congruent to the corresponding side and angles of the other triangle. Then triangles are congruent.
- ✯ If two angles of one triangle are congruent to the corresponding two angles of the other, the third angle of one will be congruent to the third of the 2nd triangle and so condition S·A·A ≅ S.A.A is true and triangles are congruent
- ✯ If S.S.S ≅ S.S.S then triangles are congruent.
- ✯ If H·S ≅ H·S i.e hypotenuse, side of Δ is congruent to corresponding hypotenuse side of other Δ. Then Δs are congruent.

EXTRA PROBLEMS#

Q:1 # Three Forces P, Q, R act along the sides of a Triangle ABC. Prove that if

$$P \operatorname{Cosec} A + Q \operatorname{Cosec} B + R \operatorname{Cosec} C = 0$$

, then the line of action of the resultant passes through the centroid of the triangle.

Sol # Centroid of Triangle is the point of concurrency of the medians of triangle and this point divide medians in ratio 2:1.

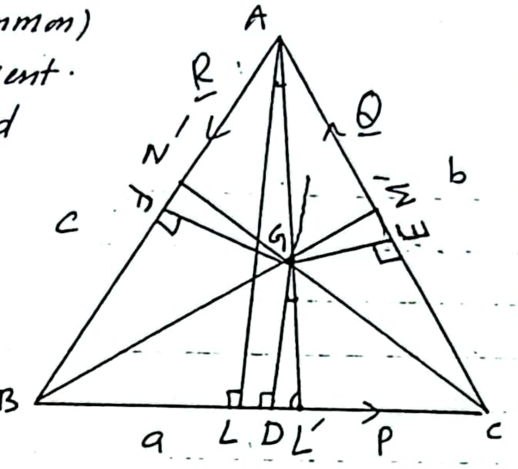
Let AL', BM', CN' be medians of ΔABC. G is the centroid of the triangle. Draw $\perp$ar GD on side BC and $\perp$ar AL on side BC. Now in the correspondence $\Delta ALL' \longleftrightarrow \Delta GDL'$, we have

$\angle ALL' \cong \angle GDL'$ (each is of 90°)

$\angle AL'L \cong \angle GL'D$ (Common)

So third angle will also be congruent.
Thus triangles are similar and we have

$$\frac{GD}{AL} = \frac{GL'}{AL'} = \frac{DL'}{LL'}$$



Now

$GL' = \frac{1}{3} AL'$

(because median AL' is divided at G in ratio 2:1).

So from

$$\frac{GD}{AL} = \frac{GL'}{AL'} = \frac{\frac{1}{3}AL'}{AL'} = \frac{1}{3}$$

$\Rightarrow GD = \frac{1}{3} AL \rightarrow ①$

But from ΔABL

$$\frac{AL}{AB} = \sin B$$

$\Rightarrow \frac{AL}{c} = \sin B \Rightarrow AL = c\sin B$

using in ①.

$GD = \frac{c\sin B}{3} \rightarrow ②$

Similarly

$GE = \frac{a\sin c}{3} \rightarrow ③$

$GF = \frac{b\sin A}{3} \rightarrow ④$

According to Varignon's theorem
Resultant will pass through G
if the sum of moments about G = 0

$\Rightarrow P\,GD + Q\,GE + R\,GF = 0$

$P\frac{c\sin B}{3} + Q\frac{a\sin C}{3} + R\frac{b\sin A}{3} = 0$

By Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$\Rightarrow a = k\sin A \quad b = k\sin B \quad c = k\sin C$

$\Rightarrow$ If $P k \sin C \sin B + Q k \sin A \sin C + R k \sin A \sin B = 0$

If $P \sin B \sin C + Q \sin C \sin A + R \sin A \sin B = 0$

If $\frac{P}{\sin A} + \frac{Q}{\sin B} + \frac{R}{\sin C} = 0$ (Dividing by $\sin A \sin B \sin C$)

If $P \operatorname{Cosec} A + Q \operatorname{Cosec} B + R \operatorname{Cosec} C = 0$ (Proved)

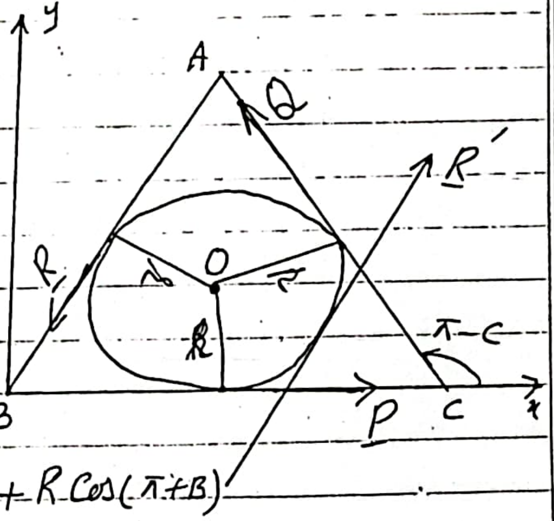
Q:2 # Three Forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC. If their resultant touches the inscribed circle, prove that

$$\frac{\cos^2 \frac{A}{2}}{P} + \frac{\cos^2 \frac{B}{2}}{Q} + \frac{\cos^2 \frac{C}{2}}{R} = 0$$

Sol # (The centre of the inscribed circle is point of Concurrency of angular bisectors of the Triangle)

Taking side BC of ΔABC along x-axis and y-axis $\perp$ar to it. Let O be the in-centre and R' be the resultant of forces.

Resolving forces along and $\perp$ar to x-axis



$$X = P\cos 0^\circ + Q\cos(\pi - C) + R\cos(\pi + B)$$

$$= P - Q\cos C - R\cos B$$

$$Y = P\sin 0^\circ + Q\sin(\pi - C) + R\sin(\pi + B)$$

$$= Q\sin C - R\sin B$$

$$R'^2 = X^2 + Y^2$$

$$R'^2 = (P - Q\cos C - R\cos B)^2 + (Q\sin C - R\sin B)^2.$$

$$= P^2 + Q^2\cos^2 C + R^2\cos^2 B - 2PQ\cos C - 2PR\cos B + 2QR\cos B\cos C + Q^2\sin^2 C + R^2\sin^2 B - 2QR\sin B\sin C$$

$$= P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B + 2QR(\cos B\cos C - \sin B\sin C)$$

$$= P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos(B+C)$$

$$= P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B + 2QR\cos(\pi - A)$$

$(\because A+B+C = \pi)$

$$R'^2 = P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B - 2QR\cos A \rightarrow ①$$

By Varignms Theorem

Sum of moments of forces about O = Moment of R' about O

$\Rightarrow$

$$R'r = Pr + Qr + Rr$$

$$R' = P + Q + R \rightarrow ②$$

By ① & ②

$$(P+Q+R)^2 = P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B - 2QR\cos A$$

$$P^2 + Q^2 + R^2 + 2PQ + 2PR + 2QR = P^2 + Q^2 + R^2 - 2PQ\cos C - 2PR\cos B - 2QR\cos A$$

$$2PQ(1+\cos C) + 2PR(1+\cos B) + 2QR(1+\cos A) = 0$$

$$PQ\cdot 2\cos^2(C/2) + PR\cdot 2\cos^2 B/2 + QR\cdot 2\cos^2 A/2 = 0$$

$$PQ\cos^2 C/2 + PR\cos^2 B/2 + QR\cos^2 \frac{A}{2} = 0$$

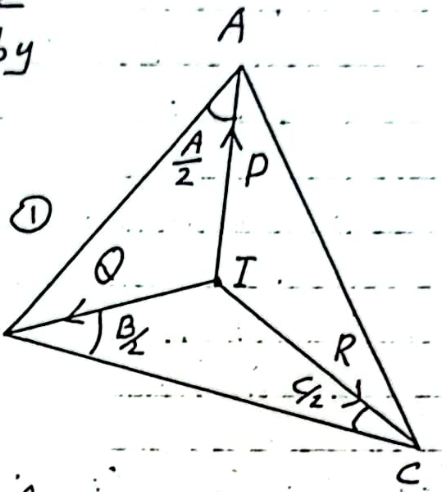
Dividing by PQR

$$\frac{\cos^2\frac{A}{2}}{P} + \frac{\cos^2 B/2}{Q} + \frac{\cos^2 C/2}{R} = 0$$

Q:3# Forces of Magnitude P, Q, R act along IA IB, IC, where I is the incentre of ΔABC, and are in equilibrium. Show that

$$P : Q : R = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$$

Sol # Since the forces are in equilibrium, therefore by Lami's Theorem



$$\frac{P}{\sin\angle BIC} = \frac{Q}{\sin\angle CIA} = \frac{R}{\sin\angle AIB} \rightarrow ①$$

From ΔIBC

$$\angle BIC = 180^\circ - \frac{B+C}{2}$$

$$= 180^\circ - \frac{180-A}{2} \qquad \because A+B+C=180^\circ$$

$$= 90^\circ + \frac{A}{2}.$$

Similarly $m\angle CIA = 90^\circ + \frac{B}{2}$

$m\angle AIB = 90^\circ + \frac{C}{2}.$

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using in ①

$$\frac{P}{\sin(90+\frac{A}{2})} = \frac{Q}{\sin(90^\circ+\frac{B}{2})} = \frac{R}{\sin(90^\circ+\frac{C}{2})}$$

$$\frac{P}{\cos\frac{A}{2}} = \frac{Q}{\cos\frac{B}{2}} = \frac{R}{\cos C/2}$$

$$\Rightarrow P : Q : R = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$$

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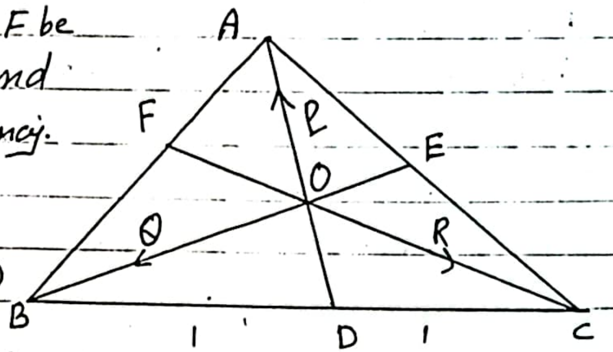
Q#4# ABC is a triangle and O is a point in its plane. Forces of magnitude P, Q, R act along OA, OB, OC respectively and are in equilibrium. Show that

(a) If O is the centroid of the triangle, then
$$P:Q:R = OA:OB:OC.$$

(b) If O is the circum-centre of the triangle, then
$$P:Q:R = \sin 2A : \sin 2B : \sin 2C.$$

Sol# Let AD, BE, CF be medians of $\triangle ABC$ and be their point of concurrency. By (λ, μ) Theorem



$\vec{Q}+\vec{R}=(1+1)\vec{OD}=2\vec{OD} \rightarrow$ ①

∵ O divides median AD in ratio 2 : 1.

∴ $AO : OD = 2:1 \Rightarrow AO = 2OD$

$\Rightarrow \vec{AO} = 2\vec{OD}$

By ①

$\vec{Q}+\vec{R} = 2\vec{OD} = \vec{AO} \rightarrow$ ②

∵ The system is in equilibrium

∴ $\vec{P}$ must balance $2\vec{OD}=\vec{AO}$

$\Rightarrow \vec{P} = \vec{OA} \Rightarrow P = OA$

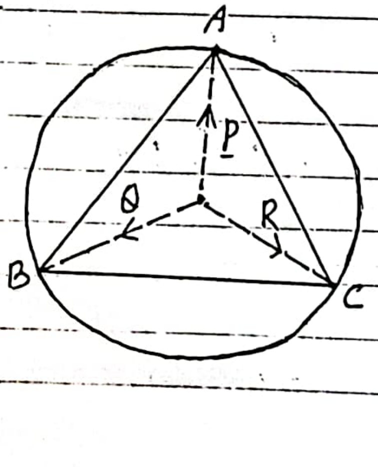
Similarly $Q = OB$ $R = OC$

Hence $P:Q:R = OA:OB:OC$.

(b) Let O be the circumcentre of the triangle ABC.

Forces acting along OA, OB, OC are in equilibrium.

∵ Angle at the centre of a circle is double the angle at the circumference made by the same arc of the circle.



$\therefore \angle BOC = 2\angle A$

Similarly

$\angle COA = 2\angle B$

$\angle AOB = 2\angle C$

$\because$ Forces are in equilibrium

$\therefore$ By Lami's Theorem

$$\frac{P}{\sin\angle BOC} = \frac{Q}{\sin\angle COA} = \frac{R}{\sin\angle AOB}$$

$$\Rightarrow \frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

$$\Rightarrow P : Q : R = \sin 2A : \sin 2B : \sin 2C.$$

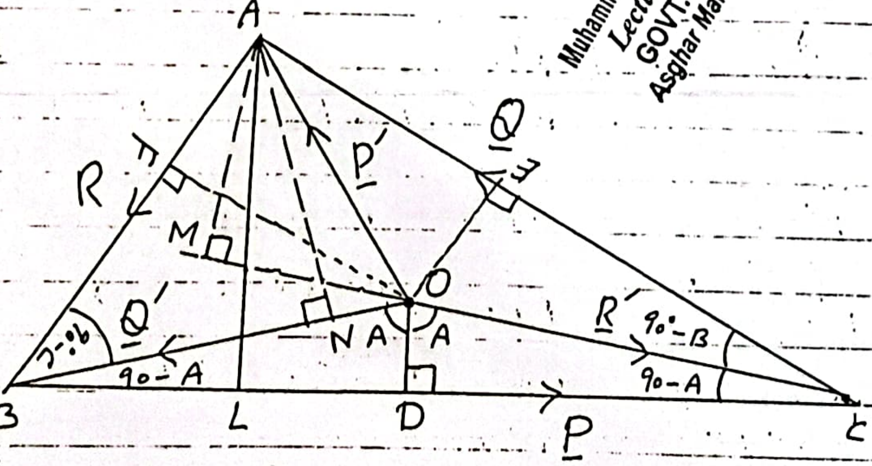
Q:5# Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, P', Q', R' act along OA, OB, OC where O is the circumcentre of the triangle ABC. Prove that if the six forces are in equilibrium, Then

$$P\cos A + Q\cos B + R\cos C = 0$$

and

$$\frac{PP'}{a} + \frac{QQ'}{b} + \frac{RR'}{c} = 0$$

Sol#



$\because$ The coplanar forces are in equilibrium

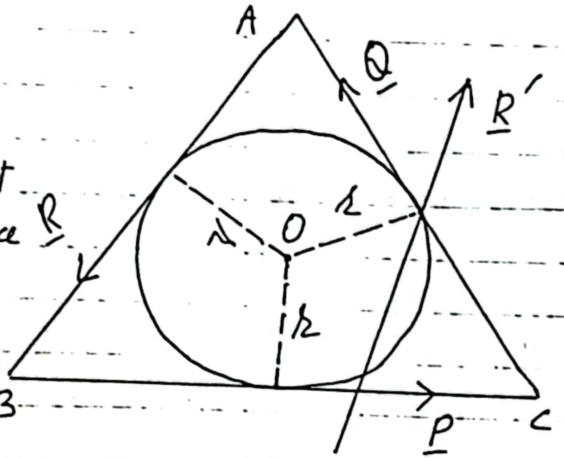
$\therefore$ The sum of the moments of the forces about any point in their plane is zero

Q:6 # Three forces P, Q, R are acting along the sides BC CA and AB of a ΔABC, then the line of action of the resultant of these forces passes through the incentre if $P+Q+R=0$

Sol # Let O be the in-centre of ΔABC.

Let R' be the resultant of the forces at distance d, from O.

Taking moments about O



$$R'd = Pr + Qr + Rr$$
$$= r(P+Q+R)$$

If $P+Q+R=0$

Then
$$R'd = 0$$
$$\Rightarrow d = 0 \quad \because R' \neq 0$$

$\Rightarrow$ the resultant passes through in-centre O

Q:7 # The Resultant of two forces P, Q is $R, 2R$ or $4R$ according as the angle between them is $\frac{\pi}{2}, \theta$ or $(\frac{\pi}{2}-\theta)$: Find the value of $\tan\theta$

Sol # When angle bet P, Q is $\frac{\pi}{2}$

$$R^2 = P^2+Q^2 \longrightarrow ①$$

When angle bet P, Q is θ

$$4R^2 = P^2+Q^2+2PQ\cos\theta \longrightarrow ②$$

When angle bet P, Q is $\frac{\pi}{2}-\theta$

$$16R^2 = P^2+Q^2+2PQ\cos(\frac{\pi}{2}-\theta)$$
$$= P^2+Q^2+2PQ\sin\theta \longrightarrow ③$$

Using ① in ② & ③, we get

$$3R^2 = 2PQ\cos\theta \longrightarrow ④$$

$$15R^2 = 2PQ\sin\theta \longrightarrow ⑤$$

Dividing ⑤ by ④

$$\tan\theta = 5$$

Q:8 # The resultant of two forces of magnitude P and Q is of magnitude R. If Q is doubled, R is doubled. If R is reversed R is again doubled: Show that

$$P^2 : Q^2 : R^2 = 2:3:2$$

Sol # Let α be the angle between P, Q: Then

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \longrightarrow ①$$

When Q is doubled, then resultant is 2R and so

$$4R^2 = P^2 + 4Q^2 + 4PQ\cos\alpha \longrightarrow ②$$

When Q is reversed (becomes -ve), R is doubled and we have

$$4R^2 = P^2 + Q^2 - 2PQ\cos\alpha \longrightarrow ③$$

Adding ① & ③

$$5R^2 = 2P^2 + 2Q^2$$

$$2P^2 + 2Q^2 - 5R^2 = 0 \longrightarrow ④$$

$③ + 2② \Rightarrow$

$$12R^2 = 3P^2 + 6Q^2$$

$$3P^2 + 6Q^2 - 12R^2 = 0 \longrightarrow ⑤$$

By ④ & ⑤ by Cross-multiplication

$$\frac{P^2}{-8+10} = \frac{Q^2}{-5+8} = \frac{R^2}{4-2}$$

$$\frac{P^2}{2} = \frac{Q^2}{3} = \frac{R^2}{2}$$

$$\Rightarrow P^2 : Q^2 : R^2 = 2:3:2$$

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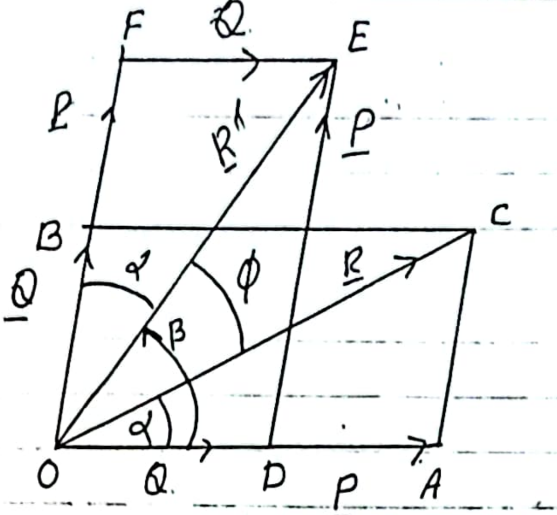

Sol # Let $\underline{R}$ be the resultant of $\underline{P}$ & $\underline{Q}$ and α be the angle between $\underline{P}$ & $\underline{R}$. When $\underline{P}$ and $\underline{Q}$ are interchanged, let $\underline{R}'$ be the new resultant making angle β with the original $\underline{P}$.

clearly the angle between $\underline{P}$ & $\underline{R}'$ will be α.

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This implies

$$\alpha+\beta=\theta$$

The angle ϕ through which resultant turns is given by

$$\phi=\beta-\alpha$$

Resolving $\underline{R}$ and $\underline{R}'$ perpendicular to OA, we have

$$R\sin\alpha = Q\sin\theta \longrightarrow ①$$

$$R'\sin\beta = P\sin\theta \longrightarrow ②$$

Also

$$R=R'=\sqrt{P^2+Q^2+2PQ\cos\theta}$$

Dividing ② by ①

$$P/Q = \frac{\sin\beta}{\sin\alpha} \qquad \because R=R'$$

By Componendo dividendo Method

$$\frac{P-Q}{P+Q} = \frac{\sin\beta-\sin\alpha}{\sin\beta+\sin\alpha} = \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta-\alpha}{2}\right)}$$

$$\frac{P-Q}{P+Q} = \cot\left(\frac{\beta+\alpha}{2}\right)\cdot\tan\left(\frac{\beta-\alpha}{2}\right)$$

$$\tan\left(\frac{\beta-\alpha}{2}\right) = \frac{P-Q}{P+Q}\tan\left(\frac{\beta+\alpha}{2}\right)$$

$$\tan\frac{\phi}{2} = \frac{P-Q}{P+Q}\tan\left(\frac{\beta+\alpha}{2}\right)$$

OR

$$\text{Tan}\,\alpha = \frac{Q\,\text{Sin}\,\theta}{P+Q\,\text{Sin}\,\theta}$$

and

$$\varphi = \theta - 2\alpha$$

$$\Rightarrow \frac{\varphi}{2} = \frac{\theta}{2} - \alpha$$

$$\Rightarrow \text{Tan}\,\frac{\varphi}{2} = \frac{\text{Tan}\,\frac{\theta}{2} - \text{Tan}\,\alpha}{1+\text{Tan}\,\frac{\theta}{2}\,\text{Tan}\,\alpha}$$

$$= \frac{\text{Tan}\,\frac{\theta}{2} - \frac{Q\,\text{Sin}\,\theta}{P+Q\,\text{Cos}\,\theta}}{1+\tan\frac{\theta}{2}\cdot\frac{Q\,\text{Sin}\,\alpha}{P+Q\,\text{Cos}\,\theta}}$$

$$= \frac{\text{Tan}\,\frac{\theta}{2}(P+Q\,\text{Cos}\,\theta) - Q\,\text{Sin}\,\theta}{P+Q\,\text{Cos}\,\theta + Q\,\text{Sin}\,\alpha\,\text{Tan}\,\frac{\theta}{2}}$$

$$= \frac{P\,\text{Tan}\,\frac{\theta}{2} + Q\left(\text{Tan}\,\frac{\theta}{2}\,\text{Cos}\,\theta - \text{Sin}\,\theta\right)}{P+Q\left(\text{Cos}\,\theta + \text{Sin}\,\alpha\,\text{Tan}\,\frac{\theta}{2}\right)}$$

$$= \frac{P\,\text{Tan}\,\theta/2 + Q\left(\frac{\text{Sin}\,\theta/2\,\text{Cos}\,\theta - \text{Cos}\,\theta/2\,\text{Sin}\,\theta}{\text{Cos}\,\theta/2}\right)}{P+Q\left(\frac{\text{Cos}\,\theta\,\text{Cos}\,\theta/2 + \text{Sin}\,\alpha\,\text{Sin}\,\theta/2}{\text{Cos}\,\theta/2}\right)}$$

$$= \frac{P\,\text{Tan}\,\theta/2 - Q\tan\theta/2}{P+\frac{Q\,\text{Cos}\,\theta/2}{\text{Cos}\,\theta/2}} = \frac{P-Q}{P+Q}\tan\theta/2$$

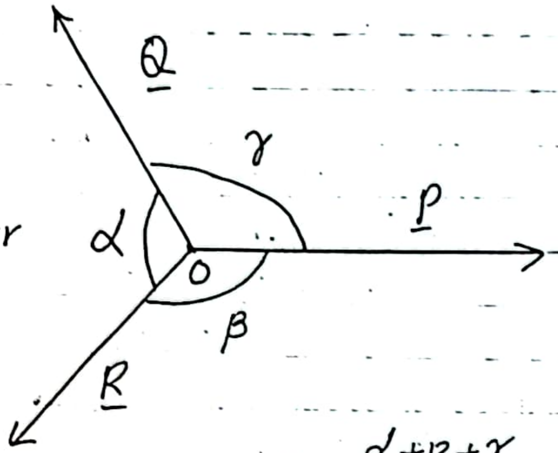
$$\text{Tan}\,\frac{\varphi}{2} = \frac{P-Q}{P+Q}\tan\frac{\theta}{2}.$$

(Proved)

Q:11 # Three forces P, Q, R in a plane act at a point. The angles between R and Q, P and R, P and Q are respectively α, β, γ. Show that the magnitude of the resultant is

$$\sqrt{P^2+Q^2+R^2+2RQ\cos\alpha+2RP\cos\beta+2PQ\cos\gamma}$$

Sol # Let the forces P, Q, R act at O and P is along x-axis. Resolving along and $\perp$ar to the x-axis



$\alpha+\beta+\gamma = 2\pi$

$$X = P\cos 0^\circ + Q\cos\gamma + R\cos(\alpha+\gamma)$$
$$= P + Q\cos\gamma + R\cos(2\pi-\beta)$$
$$= P + Q\cos\gamma + R\cos\beta$$
$$Y = P\sin 0^\circ + Q\sin\gamma + R\sin(2\pi-\beta)$$
$$= Q\sin\gamma - R\sin\beta$$

So resultant is

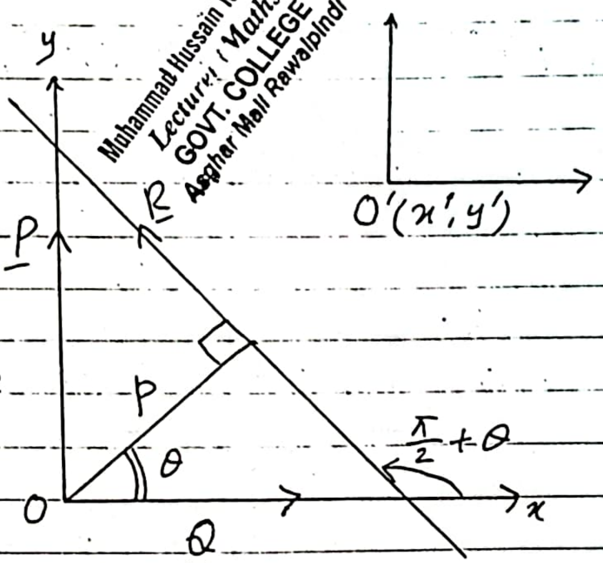
$$R' = \sqrt{X^2+Y^2}$$
$$R'^2 = (P+Q\cos\gamma+R\cos\beta)^2+(Q\sin\gamma-R\sin\beta)^2$$
$$= P^2+Q^2+R^2+2PQ\cos\gamma+2PR\cos\beta$$
$$+2QR(\cos\beta\cos\gamma-\sin\beta\sin\gamma)$$
$$= P^2+Q^2+R^2+2PQ\cos\gamma+2PR\cos\beta+2QR\cos(\beta+\gamma)$$
$$= P^2+Q^2+R^2+2PQ\cos\gamma+2PR\cos\beta+2QR\cos(2\pi-\alpha)$$
$$= P^2+Q^2+R^2+2PQ\cos\gamma+2PR\cos\beta+2QR\cos\alpha$$

Q:12 # Forces P, Q, R act along the lines $x=0$, $y=0$ and $x\cos\theta + y\sin\theta = p$. Find equation of the line of action of the resultant.

Sol# Lines are:

$x=0$ i.e y-axis $\rightarrow$ ①

$y=0$ x-axis $\rightarrow$ ②



$x\cos\theta + y\sin\theta = p \rightarrow$ ③

where p is length of ⊥ar from origin to line and θ is the inclination of ⊥ar.

$\Rightarrow$ Inclination of the line is $\frac{\pi}{2}+\theta$

Let the resultant at $O'(x', y')$ has components X and Y parallel to the axes.

∴ Resolving the forces along and ⊥ar to the x-axis, we get

$$X = Q\cos 0^\circ + P\cos\tfrac{\pi}{2} + R\cos(\tfrac{\pi}{2}+\theta)$$
$$= Q - R\sin\theta$$

$$Y = Q\sin 0^\circ + P\sin\tfrac{\pi}{2} + R\sin(\tfrac{\pi}{2}+\theta)$$
$$= P + R\cos\theta$$

Taking moment about O, we get

$$G = Rp$$

The equation of line of action of the Resultant is

$$G - xY + yX = 0$$
$$Rp - x(P+R\cos\theta) + y(Q - R\sin\theta) = 0$$
$$Rp - Px - Rx\cos\theta + yQ - yR\sin\theta = 0$$
$$Px - Qy + R(x\cos\theta + y\sin\theta - p) = 0$$
$$Px - Qy + R(0) = 0$$
$$Px - Qy = 0$$